

# Differentiable functions on modules and the equation $\text{grad}(w) = M\text{grad}(v)$

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Let  $A$  be a finite-dimensional, commutative algebra over  $\mathbb{R}$  or  $\mathbb{C}$ . The notion of  $A$ -differentiable functions on  $A$  is extended to the notion of  $A$ -differentiable functions on a finitely generated  $A$ -module  $B$ . Let  $U$  be an open, bounded and convex subset of  $B$ . When  $A$  is singly generated and  $B$  is arbitrary or  $A$  is arbitrary and  $B$  is a free module, an explicit formula for an  $A$ -differentiable functions on  $U$ , of a prescribed class of differentiability, is given in terms of real or complex differentiable functions. It appears, even in case of real algebras, that certain components of  $A$ -differentiable function are of higher differentiability than the function itself.

Let  $M$  be a constant, square matrix. Using the aforementioned formula we find the complete solution of the equation  $\text{grad}(w) = M\text{grad}(v)$ .

The boundary value problem for generalized Laplace equations  $M\nabla^2 v = \nabla^2 v M^\top$  is formulated and it is proved that for the given boundary data there exists a unique solution, for which a formula is provided.

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## 1. Introduction

Let  $M$  be a constant, square matrix. Consider the system of partial differential equations

$$\text{grad}(w) = M\text{grad}(v), \quad (1.1)$$

where  $v, w$  are real-valued functions defined on an open set  $U \subset \mathbb{R}^n$ . The first systematic treatment of this system and a complete description of solutions on convex sets was provided in [4] by Jodeit and Olver. Later, Waterhouse in [13] observed that if  $M$  has only one Jordan block corresponding to each eigenvalue, then any solution to the equation is represented by some differentiable function on a finite-dimensional, commutative algebra  $A$ . Differentiable functions on algebras (see [7, 11, 14]), which have been studied since their definition in 1891 by Georg Scheffers – a student of Sophus Lie (see [14] for more historical remarks), are not enough to provide such representation for any  $M$ . However, Waterhouse in [13] suggested that a solution in a general case could be represented by an  $A$ -differentiable function on a module. Following this suggestion we develop the theory of  $A$ -differentiable functions on modules (see §2) and prove that the analogous result for arbitrary  $M$  holds (see theorem 9.1).

Waterhouse in [14] showed that  $A$ -differentiable functions on algebras depend polynomially on some coordinates. We extend this result and prove theorems (see §6) which state the exact dependence of such functions on certain coordinates and give a precise formula in terms of real and complex differentiable functions. Not only does the §6 cover the case of differentiable functions on algebras, but also it presents the structure of any differentiable function on a free module (see theorem 7.2) and the structure of any differentiable function on arbitrary module over singly generated algebra (see theorem 8.2). Surprisingly, even in case of  $\mathbb{R}$ -algebras, certain components of an  $A$ -differentiable function are of the higher differentiability than it is assumed of the function.

Jonasson in [5] studied the equation (1.1) and posed a question, if it is true that every solution is given by a power series. We show that the answer depends on the roots of characteristic polynomial of  $M$ . If there are no real roots, then the answer is positive and, conversely, if the opposite case occurs, then the answer is negative (see theorem 9.2 and §4).

When analysing the equation (1.1), the generalized Laplace equations

$$M\nabla^2 v = \nabla^2 v M^\top \quad (1.2)$$

appear naturally, as these are exactly the integrability conditions for (1.1). We are thus interested in the boundary value problem for equations (1.2), to which §10 is devoted. The connection with  $A$ -differentiable functions on modules (see theorem 3.8) and the structure theorem allows us to provide appropriate formulation of boundary value problem (see theorem 10.2) such that (1.2) have a unique solution for a given boundary data. Moreover, there exists continuous linear operator that maps boundary data to the unique solution.

There are several questions which arise. Let us enumerate some of them.

- (i) How does an  $A$ -differentiable function between arbitrary  $A$ -modules look?
- (ii) How to generalize the notion of  $A$ -differentiability to non-commutative algebras so that any  $A$ -analytic function is  $A$ -differentiable and the set of  $A$ -differentiable functions forms an algebra?
- (iii) How to solve equation  $\text{grad}(w) = M\text{grad}(v)$  when  $M$  is a non-constant matrix?
- (iv) What are other partial differential equations such that the space of solutions forms an algebra? What are the conditions for such property? When is such algebra semisimple? When is it finitely generated?

The paper should be understandable for readers without any previous knowledge of the subject.

## 2. Differentiable functions on modules

We begin with a definition of differentiability of functions on modules. Throughout the paper  $A$  denotes a finite dimensional *commutative*  $\mathbb{F}$ -algebra with unit over the field  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ .  $B, C$  stand for finitely generated  $A$ -modules.  $U \subset B$  is an open

set. We shall denote by  $\|\cdot\|$  a norm. Usually it can be chosen arbitrarily, otherwise we shall specify what conditions we want a norm to satisfy.

Recall that module  $B$  over  $A$  is finitely generated if surjective module homomorphism  $\eta: A^n \rightarrow B$  exists. For finite dimensional  $A$  this happens if and only if  $B$  is finite dimensional as a vector space.

If  $B$  be a finitely generated free module, we may choose certain  $A$ -basis  $b_1, \dots, b_n$  for  $B$ . For  $x = \sum_{i=1}^n x_i b_i$ ,  $y = \sum_{i=1}^n y_i b_i$  and  $C = [c_{ij}]_{i,j=1,\dots,n} \in M_{n \times n}(A)$  we write  $x^\top y = \sum_{i=1}^n x_i y_i$  and  $Cy = \sum_{j=1}^n c_{ij} y_j b_i$ .

**Definition 2.1.** A function  $f: U \rightarrow C$  is said to be  $A$ -differentiable if a map  $Df: U \times B \rightarrow C$  exists, such that

- (i)  $\lim_{y \rightarrow 0} \frac{1}{\|y\|} \|f(x+y) - f(x) - Df(x)(y)\| = 0$ ,
- (ii)  $Df(x)(ay) = aDf(x)(y)$  for all  $x \in U$ ,  $y \in B$  and  $a \in A$ .

**Remark 2.2.** If  $f$  is differentiable as a function of its real variables and the linear derivative  $Df: U \times B \rightarrow C$  satisfies  $Df(x)(ay) = aDf(x)(y)$  for all  $x \in U$ ,  $y \in B$  and  $a \in A$ , then  $f$  is  $A$ -differentiable.

**Remark 2.3.** As  $B, C$  and  $A$  are finite dimensional, all norms on these spaces are equivalent. Thus the condition of  $A$ -differentiability does not depend on the choice of the norm.

Recall definition of differentiability on algebras (see [13], [14]) – a function  $f: U \rightarrow A$  on  $U \subset A$  is said to be  $A$ -differentiable if for every  $x \in A$  there exists a limit

$$\lim_{\substack{y \rightarrow x \\ y-x \in G(A)}} \frac{f(y) - f(x)}{y - x},$$

where  $G(A)$  denotes the group of invertible elements in  $A$ . The above definition of  $A$ -differentiability agrees with the definition 2.1 if we put  $B = C = A$ . Indeed, as is proved in [14], if the above limit exists, then  $f$  is differentiable as a function of its real variables. The derivative at a point  $x \in U$  is given by a multiplication by a certain element  $f'(x) \in A$ , so it is  $A$ -linear. The converse is trivial.

We shall indicate that the condition stated in the definition constitutes an equivalent set of partial differential equations for  $f$  seen as a function of real variables. We write down these equations for  $A$ -valued functions. Let  $c_1, \dots, c_t$  denote the  $\mathbb{F}$ -basis of  $B$  for the moment. Let  $e_1, \dots, e_k$  denote  $\mathbb{F}$ -basis of  $A$ , and let  $e_i c_j = \sum_{s=1}^t \alpha_{ij}^s c_s$ . Our condition is equivalent to the equations:

$$\sum_{s=1}^t \alpha_{ij}^s Df(\cdot)(c_s) = e_i Df(\cdot)(c_j).$$

These equations are reduced to ordinary Cauchy-Riemann equations when  $A = \mathbb{C}$ , treated as  $\mathbb{R}$ -algebra, and  $B = A$ . When  $B = \mathbb{C}^n$  and  $A = \mathbb{C}$ , then these are the conditions for a function of several complex variables to be holomorphic.

**Lemma 2.4.** *Let  $k \geq 1$ . Let  $f: U \rightarrow C$  be a  $C^k$  function that is  $A$ -differentiable. Then for any  $x_1, \dots, x_{k-1} \in B$  the function  $D^{k-1}f(\cdot)(x_1, \dots, x_{k-1}): U \rightarrow C$  is again  $A$ -differentiable. Moreover*

$$D^k f(\cdot)(ax_1, \dots, x_k) = aD^k f(\cdot)(x_1, \dots, x_k)$$

for all  $x_1, \dots, x_k \in B$  and all  $a \in A$ .

*Proof.* Let  $g(\cdot) = D^{k-1}f(\cdot)(x_1, \dots, x_{k-1})$ . We have

$$\begin{aligned} D^k f(\cdot)(ax_1, \dots, x_k) &= D^{k-1}(Df(\cdot)(ax_1))(x_2, \dots, x_k) = \\ &= D^{k-1}(aDf(\cdot)(x_1))(x_2, \dots, x_k) = aD^k f(\cdot)(x_1, \dots, x_k). \end{aligned}$$

Thus  $Dg(\cdot)(ay) = aDg(\cdot)(y)$ .  $\square$

**Lemma 2.5.** *Let  $B, C, D$  be  $A$ -modules and let  $U \subset B$  be open set. Let  $f: U \rightarrow D$ . Let  $\pi: C \rightarrow B$  be an  $A$ -linear surjection. Then*

- (i)  *$f$  is  $A$ -differentiable on  $U$  if and only if  $f \circ \pi$  is  $A$ -differentiable on  $\pi^{-1}(U)$ ,*
- (ii)  *$f \in C^k(U)$  if and only if  $f \circ \pi \in C^k(\pi^{-1}(U))$ ,*
- (iii)  *$f \in C^k(\overline{U})$  if and only if  $f \circ \pi \in C^k(\overline{\pi^{-1}(U)})$ ,*

*Proof.* One needs to verify that  $Df$  exists and is  $A$ -linear if  $D(f \circ \pi)$  exists and is  $A$ -linear. Let us show that  $Df$  exists.

For some linear subspace  $V \subset C$ , we have  $C = V \oplus \ker \pi$ , as vector spaces. On  $C$  we may introduce a norm by the formula

$$\|r + s\| = \|\pi(r)\|_B + \|s\|_C$$

for  $r \in V$  and  $s \in \ker \pi$  and some norms  $\|\cdot\|_B$  on  $B$  and  $\|\cdot\|_C$  on  $\ker \pi$ .

For a given  $x \in U$ ,  $y \in B$  choose  $t \in \pi^{-1}(x)$  and  $r \in \pi^{-1}(y) \cap V$  such that  $\|y\|_B = \|\pi(r)\|_B = \|r\|$ . Since  $\|r\| \rightarrow 0$  if  $\|y\|_B \rightarrow 0$ , we have

$$\begin{aligned} \frac{1}{\|y\|_B} \|f(x+y) - f(x) - D(f \circ \pi)(t)(r)\| &= \\ = \frac{1}{\|r\|} \|f(\pi(t+r)) - f(\pi(t)) - D(f \circ \pi)(t)(r)\| &\xrightarrow{y \rightarrow 0} 0. \end{aligned}$$

Hence  $f$  is differentiable and  $Df(x)(y) = D(f \circ \pi)(t)(r)$ .  $A$ -linearity follows from  $A$ -linearity of  $D(f \circ \pi)$  and  $\pi$ . Indeed, for  $a \in A$  we write  $ar = v_a + k_a$ , with  $v_a \in V$ ,  $k_a \in \ker \pi$ . Then  $\pi(ar) = \pi(v_a)$  so  $v_a \in \pi^{-1}(ay) \cap V$ . Thus

$$Df(x)(ay) = Df(\pi(t))(\pi(v_a)) = D(f \circ \pi)(t)(v_a) = D(f \circ \pi)(t)(ar - k_a) = aDf(x)(y).$$

Let us prove that  $f$  is of class  $C^k$  if  $f \circ \pi$  is of class  $C^k$ . We shall proceed inductively. Assume that we have shown that for some  $l \leq k-1$  we have

$$D^l f(x)(y_1, \dots, y_l) = D^l(f \circ \pi)(t)(r_1, \dots, r_l) \quad (2.1)$$

for  $t \in \pi^{-1}(x) \cap V$  and  $r_i \in \pi^{-1}(y_i) \cap V$ ,  $i = 1, \dots, l$ . Then, by lemma 2.4, for given  $y_1, \dots, y_l$  function  $D^l f(\pi(\cdot))(y_1, \dots, y_l)$  is  $A$ -differentiable. The previous argument shows that  $D^l f(\cdot)(y_1, \dots, y_l)$  is  $A$ -differentiable and

$$\begin{aligned} D^{l+1} f(x)(y_1, \dots, y_l, y_{l+1}) &= D(D^l f(\pi(\cdot))(y_1, \dots, y_l))(t)(r_{l+1}) = \\ &= D^{l+1} (f \circ \pi)(t)(r_1, \dots, r_{l+1}) \end{aligned}$$

for some  $r_{l+1} \in \pi^{-1}(y_{l+1}) \cap V$ . It follows that  $f$  is  $k$ -times differentiable if  $f \circ \pi$  is.

Assume that  $f \circ \pi \in C^k(\pi^{-1}(U))$ . Then, by equation (2.1),  $f \in C^k(U)$ . For if,  $(x_n)_{n=1}^\infty \subset U$  converges to  $x$ , then choosing  $t_n \in \pi^{-1}(x_n) \cap V$ , and  $t \in \pi^{-1}(x) \cap V$  we have  $\|t_n - t\|_B = \|\pi(x_n - x)\|_B = \|x_n - x\|$ , so  $t_n$  converges to  $t$ .

Assume that  $f \circ \pi \in C^k(\overline{\pi^{-1}(U)})$ . Then again the equation (2.1) allows us to extend derivatives of  $f$  continuously on  $\overline{U}$ , as we have  $\pi^{-1}(\overline{U}) = \overline{\pi^{-1}(U)}$ .  $\square$

### 2.1. Equivalent characterizations of $A$ -differentiability

We shall deal now with  $A$ -valued forms on  $U$ . As previously, we shall denote by  $e_1, \dots, e_k$  a basis of  $A$  over  $\mathbb{F}$ . By  $x_1, \dots, x_k$  we now denote the corresponding coordinate functions. If  $x: U \rightarrow A$  is a joint  $A$ -coordinate function, then  $x = \sum_{i=1}^k x_i e_i$ , so that

$$dx = \sum_{i=1}^k e_i dx_i.$$

It is clear that Poincaré's lemma and standard rules for differentiating wedge-product of two forms hold.

**Theorem 2.6.** *Assume that  $U \subset A$  is open and simply connected. Let  $f: U \rightarrow A$  be a  $C^1$  function. The following conditions are equivalent*

- (i)  $f$  is  $A$ -differentiable,
- (ii) form  $f dx$  is closed,
- (iii)  $\int_\gamma f dx = 0$  for any smooth closed curve  $\gamma$  in  $U$ ,
- (iv) there exists  $A$ -differentiable  $C^2$  function  $g: U \rightarrow A$  such that

$$Dg(b)(x) = xf(b)$$

for all  $b \in U$  and  $x \in A$ .

*Proof.* Let us first see what the condition (ii) means. The form  $f dx$  is closed if and only if  $dx \wedge df = 0$ . That is

$$\begin{aligned} 0 &= \left( \sum_{l=1}^k e_l dx_l \right) \wedge \left( \sum_{i=1}^k Df(\cdot)(e_i) dx_i \right) = \\ &= \sum_{0 \leq i < l \leq k} (e_l Df(\cdot)(e_i) - e_i Df(\cdot)(e_l)) dx_l \wedge dx_i. \end{aligned}$$

This is equivalent to  $e_l Df(\cdot)(e_i) = e_i Df(\cdot)(e_l)$ . By bilinearity of both sides we have equivalently

$$y Df(\cdot)(x) = x Df(\cdot)(y) \quad (2.2)$$

for all  $x, y \in A$ . Let  $a \in A$ . Then

$$y Df(\cdot)(ax) = ax Df(\cdot)(y) = a(y Df(\cdot)(x)) = y(a Df(\cdot)(x)). \quad (2.3)$$

Letting  $y$  to be the unit of  $A$ , we see that if (ii) is satisfied, then so is (i). Conversely, knowing that (i) is true, then (2.3) holds and so does (2.2). This condition is equivalent to (ii).

If we know that there is an  $A$ -differentiable  $g$  is as in (iv), then by lemma 2.4, we have

$$x Df(b)(ay) = D^2 g(b)(x, ay) = a D^2 g(b)(x, y) = x(a Df(b)(y)),$$

thus  $Df(b)$  is  $A$ -linear.

Assume that the form  $f dx$  is closed. By Poincaré's lemma there exists a  $C^2$  function  $g: U \rightarrow A$  such that  $dg = f dx$ . That is

$$\sum_{i=1}^k Dg(\cdot)(e_i) dx_i = \sum_{i=1}^k f e_i dx_i.$$

This condition is equivalent to  $Dg(\cdot)(x) = x f(\cdot)$ . In particular  $g$  is  $A$ -differentiable.

Assume that condition (iii) is satisfied by  $f: U \rightarrow A$ . Choose any disc  $D \subset U$ , let  $\gamma$  be its boundary. Then by Stokes' theorem

$$\int_D df \wedge dx = \int_D d(f dx) = \int_\gamma f dx = 0.$$

Since  $D$  was arbitrary and  $df \wedge dx$  is continuous, we conclude that (ii) holds.

Assume that (ii) is fulfilled. Take any smooth closed curve  $\gamma: [0, 1] \rightarrow U$ . By Whitney's approximation theorem (see [9], theorem 6.21) and by simple connectedness of  $U$ ,  $\gamma$  is a boundary of some smooth surface  $D \subset U$ . Again by Stokes' theorem, the integral

$$\int_\gamma f dx = \int_D d(f dx)$$

vanishes. □

Below,  $B$  denotes a free module over  $A$ . If  $U \subset B$  and  $f: U \rightarrow B$  then we write  $f = \sum_{i=1}^n f_i b_i$ , where  $b_1, \dots, b_n$  is an  $A$ -basis of  $B$ . We shall consider  $A$ -valued forms on  $U$ . In particular  $x_i$  denotes  $A$ -valued  $i$ -th coordinate function on  $U$  and  $dx_i$  the corresponding form.

**Proposition 2.7.** *Let  $U \subset B$  be open and simply connected. Let  $f: U \rightarrow B$  be of class  $C^1$ . The following are equivalent:*

- (i)  *$f$  is  $A$ -differentiable and such that  $Df(\cdot)$  satisfies*

$$Df_i(\cdot)(b_j) = Df_j(\cdot)(b_i),$$

*for all  $i, j = 1, \dots, n$ ,*

(ii) there exists  $C^2$ ,  $A$ -differentiable function  $g: U \rightarrow A$ , such that

$$Dg(\cdot)(x) = x^\top f(\cdot).$$

*Proof.* Define one-form

$$h = \sum_{i=1}^n f_i dx_i.$$

Then  $dh = 0$ . Indeed, by  $A$ -linearity and the assumption we have

$$\begin{aligned} dh &= \sum_{i,s=1}^n \sum_{j,r=1}^k e_j Df_i(\cdot)(e_r b_s) dx_{rs} \wedge dx_{ji} = \sum_{i,s=1}^n Df_i(\cdot)(b_s) dx_s \wedge dx_i = \\ &= \sum_{0 \leq i < s \leq n} (Df_i(\cdot)(b_s) - Df_s(\cdot)(b_i)) dx_s \wedge dx_i = 0. \end{aligned}$$

By Poincaré's lemma there exists a  $C^2$  function  $g: U \rightarrow A$  such that  $dg = h$ . That is

$$\sum_{i=1}^n \sum_{j=1}^k e_j f_i(\cdot) dx_{ji} = \sum_{i=1}^n \sum_{j=1}^k Dg(\cdot)(e_j b_i) dx_{ji},$$

thus  $Dg(\cdot)(e_j b_i) = e_j f_i(\cdot)$ . Since this holds for all  $j = 1, \dots, k$ , we have

$$Dg(\cdot)(y b_i) = y f_i(\cdot)$$

for all  $y \in A$  and in consequence  $Dg(\cdot)(x) = x^\top f(\cdot)$  for all  $x \in B$ .

To prove the converse, let  $g$  be as in (ii). Then  $D^2g(\cdot)(x, y) = x^\top Df(\cdot)(y)$ . By lemma 2.4  $Df(\cdot)$  is  $A$ -linear. Further, since  $D^2g(\cdot)$  is symmetric, we have

$$x^\top Df(\cdot)(y) = y^\top Df(\cdot)(x).$$

Taking  $x = b_i$  and  $y = b_j$  we obtain  $Df_i(\cdot)(b_j) = Df_j(\cdot)(b_i)$ . □

For any  $b \in B$  define  $t_j^b: A \rightarrow B$  by the formula

$$t_j^b(a) = ab_j + b.$$

The following proposition tells, roughly speaking, that a differentiable function is  $A$ -differentiable if and only if it is  $A$ -differentiable on  $A$ -lines.

**Proposition 2.8.** *Let  $U \subset B$  be open. Let  $f: U \rightarrow A$  be a differentiable function. The following conditions are equivalent*

- (i)  $f$  is  $A$ -differentiable,
- (ii) for any  $x \in B$  and  $j = 1, \dots, n$ ,

$$f \circ t_j^x: (t_j^x)^{-1}(U) \rightarrow A$$

*is  $A$ -differentiable.*

*Proof.* If  $f$  is  $A$ -differentiable, then  $f \circ t_j^x$  is also  $A$ -differentiable, since  $t_j^x$  is given by  $A$ -linear function plus some constant.

Assume that  $f \circ t_j^b$  is  $A$ -differentiable for any  $b \in B$  and any  $j = 1, \dots, n$ . Choose  $b \in U$  and  $x \in B$ ,  $x = \sum_{i=1}^n a_i b_i$ , and  $a \in A$ . Then

$$\begin{aligned} Df(b)(a \sum_{i=1}^n a_i b_i) &= \sum_{i=1}^n Df(b)(a a_i b_i) = \sum_{i=1}^n Df(t_i^b(0))(D t_i^b(0)(a a_i)) \\ &= \sum_{i=1}^n D(f \circ t_i^b)(0)(a a_i) = a \sum_{i=1}^n D(f \circ t_i^b)(0)(a_i) = a Df(b)(\sum_{i=1}^n a_i b_i). \end{aligned}$$

That is  $Df(b)$  is  $A$ -linear.  $\square$

**Corollary 2.9.** *Assume that  $U \subset B$  open set such that for  $j = 1, \dots, n$  and any  $b \in B$  the sets  $(t_j^b)^{-1}(U)$  are simply connected. Let  $f: U \rightarrow B$  be a  $C^1$  function. The following conditions are equivalent*

- (i)  $f$  is  $A$ -differentiable,
- (ii) for  $j = 1, \dots, n$  and any  $b \in B$  form  $(f \circ t_j^b)dx$  is closed,
- (iii) for  $j = 1, \dots, n$  and any  $b \in B$   $\int_{\gamma} (f \circ t_j^b)dx = 0$  for any smooth closed curve  $\gamma$ .

*Proof.* Conditions (i)-(iii) are equivalent by theorem 2.6 and proposition 2.8.  $\square$

**Remark 2.10.**  $A$ -differentiability is a local condition. It is thus equivalent to the fact that conditions (ii) and (iii) hold locally. The assumption about simple connectedness is not required in this case.

## 2.2. $A$ -continuity

The condition (iii) can be stated for continuous functions, not necessarily differentiable. For example, when  $A = \mathbb{R}$ , then it is satisfied by all continuous functions. If  $B = A$ , then it implies the existence of a primitive function, which is  $A$ -differentiable. Moreover, when  $A$  is a  $\mathbb{C}$ -algebra, then every function which satisfies this condition is actually analytic.

**Proposition 2.11.** *Assume that  $U \subset A$  is simply connected. Let  $f: U \rightarrow B$  be a continuous function. The following conditions are equivalent*

- (i)  $\int_{\gamma} f dx = 0$  for any smooth closed curve  $\gamma$ ,
- (ii) there exists  $A$ -differentiable  $C^1$  function  $g: U \rightarrow A$ , such that  $Dg(b)(x) = xf(b)$  for all  $b \in U$  and  $x \in B$ .

*Proof.* Choose a point  $b_0 \in U$ . Let  $b \in U$  and let  $\gamma$  be a smooth curve with  $\gamma(0) = b_0$  and  $\gamma(1) = b$ . Define  $g(b) = \int_{\gamma} f dx$ . Definition of  $g$  is invariant of the choice of  $\gamma$



as a consequence of the condition (i). We compute

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1}{\|x\|} \|g(b+x) - g(b) - xf(b)\| = \\ &= \lim_{x \rightarrow 0} \frac{1}{\|x\|} \left\| \int_{[b, b+x]} f dx - xf(b) \right\| \leq \\ &\leq \lim_{x \rightarrow 0} \frac{1}{\|x\|} \int_0^1 \|(f(b+tx) - f(b))x\| dt = 0. \end{aligned}$$

This means exactly that  $Dg(b)(x) = xf(b)$ .

Conversely, if (ii) holds then, as in theorem 2.6,  $g$  satisfies  $dg = f dx$ . Choose a smooth curve  $\gamma$ . Again as in theorem 2.6 we may assume that  $\gamma$  is a boundary of a smooth surface  $D$ . Then by Stokes' theorem and easy approximation argument

$$\int_{\gamma} f dx = \int_{\gamma} dg = 0.$$

□

**Definition 2.12.** Let  $A^n$  be a free module and  $B$  be finitely generated  $A$ -module, let  $\eta: A^n \rightarrow B$  be surjective. Let  $U \subset B$  be an open set. A function  $f: U \rightarrow A$  is called *A-continuous*, if it is continuous and such that every point in  $U$  admits a neighbourhood  $V \subset U$  such that

$$\int_{\gamma} (f \circ \eta \circ t_j^b) dx = 0$$

for any smooth closed curve  $\gamma \subset (\eta \circ t_j^b)^{-1}(V)$ , any  $j = 1, \dots, n$  and any  $b \in B$ .

**Remark 2.13.** Assume that  $f$  is of class  $C^1$ . Then lemma 2.5, corollary 2.9 imply that  $f: U \rightarrow A$  is  $A$ -continuous if and only if it is  $A$ -differentiable.

### 3. Components of $A$ -differentiable functions

We aim to characterize  $\mathbb{F}$ -valued functions defined on arbitrary finitely generated  $A$ -module  $B$  which arise as components of  $A$ -differentiable functions on  $B$ . That is, we shall specify differential conditions for a function  $v: U \rightarrow \mathbb{F}$  which are satisfied if and only if  $v = \psi(f)$  for some  $\mathbb{F}$ -linear functional  $\psi: A \rightarrow \mathbb{F}$  and some  $A$ -differentiable function  $f$  defined on an open, simply connected set  $U \subset B$ . Such  $v$  will be called a *component function*. If  $B = A$ , then the conditions are described in [13]. We follow essentially the same lines as therein, except the main theorem; we generalize the results from [13].

#### 3.1. Algebraic preliminaries

Before we come to main point of this section, we shall now recall some algebraic notions. This part comprises a slight extension of some facts from [13].

**Definition 3.1.** A commutative algebra  $A$  is called a *Frobenius algebra* if there is a linear functional  $\phi: A \rightarrow \mathbb{F}$  such that the bilinear form  $(x, y) \mapsto \phi(xy)$  is nondegenerate.

Alternatively, one could say that for every linear functional on  $A$  there exists unique  $y \in A$  such that the functional is of the form  $x \mapsto \phi(xy)$ .

**Example 3.2.** Let  $A = \mathbb{F}[x]/(P(x))$  be a quotient algebra of some polynomial  $P$ . Then  $A$  is a Frobenius algebra. We leave the details to the reader.

Next we consider linear and bilinear functions on  $A$ -modules.

**Lemma 3.3.** *Let  $A$  be a Frobenius algebra and  $B$  be a finitely generated free module over  $A$ . Let  $K: B \rightarrow \mathbb{F}$  be a linear functional. Then there exists a unique  $c \in B$  such that  $K(x) = \phi(x^\top c)$ .*

*Proof.* Function  $x \mapsto K(xb_i)$  is linear, thus it is equal to  $\phi(xc_i)$  for some unique  $c_i \in A$ . Then

$$K\left(\sum_{i=1}^n x_i b_i\right) = \sum_{i=1}^n \phi(x_i c_i) = \phi(x^\top c).$$

□

**Lemma 3.4.** *Let  $A$  and  $B$  be as in the preceding lemma. Let  $L: B \times \dots \times B \rightarrow \mathbb{F}$  be a multilinear form such that  $L(x_1, \dots, ax_l, \dots, x_r) = L(x_1, \dots, ax_k, \dots, x_r)$  for all  $x_1, \dots, x_r \in B$ , all  $a \in A$  and all  $k, l = 1, \dots, r$ . Then there exist unique elements  $c_{i_1, \dots, i_r} \in A$ ,  $i_1, \dots, i_r = 1, \dots, n$  such that*

$$L(x^1, \dots, x^r) = \phi\left(\sum_{i_1, \dots, i_r=1}^n x_{i_1}^1 \cdots x_{i_r}^r c_{i_1, \dots, i_r}\right)$$

for all  $x^i = \sum_{j=1}^n x_j^i b_j$ ,  $i = 1, \dots, r$ .

*Proof.* Function  $x \mapsto L(xb_{i_1}, b_{i_2}, \dots, b_{i_r})$  is linear, thus it is equal to  $\phi(xc_{i_1, \dots, i_r})$  for some unique  $c_{i_1, \dots, i_r} \in A$ . We have

$$L(a_1 b_{i_1}, a_2 b_{i_2}, \dots, a_r b_{i_r}) = L(a_1 \cdots a_r b_{i_1}, b_{i_2}, \dots, b_{i_r}).$$

Therefore

$$L\left(\sum_{i_1=1}^n x_{i_1}^1 b_{i_1}, \sum_{i_2=1}^n x_{i_2}^2 b_{i_2}, \dots, \sum_{i_r=1}^n x_{i_r}^r b_{i_r}\right) = \sum_{i_1, \dots, i_r=1}^n \phi(x_{i_1}^1 \cdots x_{i_r}^r c_{i_1, \dots, i_r}).$$

□

**Example 3.5.** If  $r = 2$ , then  $(c_{i_1, i_2})_{i_1, i_2=1, \dots, n}$  form a matrix  $C \in M_{n \times n}(A)$ . Then  $L$  is given by  $L(x, y) = \phi(x^\top C y)$ .

### 3.2. Components and generalized Laplace equations

**Lemma 3.6.** *Let  $v: U \rightarrow \mathbb{F}$  be a component function of an  $A$ -differentiable  $C^2$  function  $f: U \rightarrow A$ . Then*

$$D^2 v(b)(ax, y) = D^2 v(b)(x, ay)$$

for all  $x, y \in B$  and  $a \in A$ .

*Proof.* By assumption,  $v = \psi(f)$ . By lemma 2.4 we have

$$D^2v(b)(ax, y) = \psi(D^2f(b)(ax, y)) = \psi(D^2f(b)(x, ay)) = D^2v(b)(x, ay).$$

□

**Definition 3.7.** Let  $U \subset B$ . We say that  $U$  is *short-path connected*, if there exists function  $h: [0, \infty) \rightarrow \mathbb{R}$ ,  $h(0) = 0$ , continuous in 0, such that for any points  $x, y \in U$  there exists path  $\gamma \subset U$  such that its length  $|\gamma|$  satisfies  $|\gamma| \leq h(\|x - y\|)$ .

**Theorem 3.8.** Suppose that  $A$  is a Frobenius algebra and let  $B$  be an  $A$ -module. Assume that  $U \subset B$  is open and simply connected. Let  $t \geq 0$  be a natural number. Let  $v: U \rightarrow \mathbb{F}$  be a  $C^{2+t}$  function such that

$$D^2v(b)(ax, y) = D^2v(b)(x, ay)$$

for all  $b \in U$  and  $x, y \in B$  and  $a \in A$ . Then  $v = \phi(f)$  for some  $A$ -differentiable function  $f$  of class  $C^{2+t}$ . Such  $f$  is uniquely determined by  $v$ , up to a constant. If  $U$  is short-path connected, then if  $v \in C^{2+t}(\overline{U})$ , then  $f \in C^{2+t}(\overline{U})$ .

**Lemma 3.9.** The statement of theorem 3.8 is true for free modules.

*Proof.* By lemma 3.3 we see that for all  $b \in U$  we have  $Dv(b)(x) = \phi(x^\top g(b))$  for uniquely determined  $g(b)$ . By lemma 3.4 we have  $D^2v(b)(x, y) = \phi(x^\top C(b)y)$  for uniquely determined  $C(b) \in M_{n \times n}(A)$ .

We shall show that  $g: U \rightarrow B$  is  $A$ -differentiable and  $Dg(b)(y) = C(b)y$ . For all  $x \in B$  we have

$$\lim_{y \rightarrow 0} \frac{1}{\|y\|} (Dv(b+y)(x) - Dv(b)(x) - D^2v(b)(x, y)) = 0.$$

By our previous observations

$$\lim_{y \rightarrow 0} \phi \left( x^\top \left( \frac{1}{\|y\|} (g(b+y) - g(b) - C(b)y) \right) \right) = 0.$$

That means, by lemma 3.3, that the quotients  $\frac{1}{\|y\|} (g(b+y) - g(b) - C(b)y)$  converge weakly to zero. Since  $B$  is finite dimensional, as a vector space over  $\mathbb{F}$ , they converge in norm. In conclusion  $g$  is differentiable and  $Dg(b)(y) = C(b)y$ . Thus the derivative is  $A$ -linear.

Observe that  $C(\cdot)$  is continuous on  $U$ . Indeed, as  $D^2v(\cdot)(x, y)$  is continuous for all  $x, y$ , if  $b_n \rightarrow b$  then

$$\phi(x^\top C(b_n)y) = D^2v(b_n)(x, y) \rightarrow D^2v(b)(x, y) = \phi(x^\top C(b)y). \quad (3.1)$$

Again, by lemma 3.3,  $C(b_n)y$  converges to  $C(b)y$  for all  $y$ . In consequence  $C(b_n)$  converges to  $C(b)$ . Hence  $g$  is of class  $C^1$ .

Theorem 2.7 tells us that there exists  $A$ -differentiable  $f: U \rightarrow A$  such that  $Df(b)(x) = x^\top g(b)$  for all  $b \in U$  and  $x \in B$ . Such  $f$  is unique up to a constant. We have

$$D\phi(f(b))(x) = \phi(Df(b)(x)) = \phi(x^\top g(b)) = Dv(b)(x).$$

Thus  $\phi(f)$  and  $v$  differ only by a constant. Adding suitable constant to  $f$  we get  $v = \phi(f)$ .

We have to show that  $f \in C^{2+t}(U)$  if  $v \in C^{2+t}(U)$ . For all  $2 \leq k \leq t+2$  the form

$$(x_1, \dots, x_k) \mapsto D^k v(\cdot)(x_1, \dots, x_k)$$

satisfies the assumptions of lemma 3.4. Indeed, as  $D^2 v(\cdot)$  does, we have<sup>1</sup>

$$\begin{aligned} D^k v(\cdot)(x_1, \dots, ax_l, \dots, x_k) &= D^{k-2}(D^2 v(\cdot)(ax_l, x_p))(x_1, \dots, \hat{x}_l, \dots, \hat{x}_p, \dots, x_k) = \\ &= D^{k-2}(D^2 v(\cdot)(x_l, ax_p))(x_1, \dots, \hat{x}_l, \dots, \hat{x}_p, \dots, x_k) = D^k v(\cdot)(x_1, \dots, ax_p, \dots, x_k) \end{aligned}$$

for all  $l < p \leq k$ . Thus there exist  $c_{i_1, \dots, i_k}(b) \in A$ ,  $i_1, \dots, i_k = 1, \dots, n$ , such that

$$D^k v(b)(x^1, \dots, x^k) = \phi \left( \sum_{i_1, \dots, i_k=1}^n x_{i_1}^1 \cdots x_{i_k}^k c_{i_1, \dots, i_k}(b) \right)$$

Assume that we have shown that  $f \in C^k(U)$  for some  $k < 2+t$  and

$$D^k f(b)(x^1, \dots, x^k) = \sum_{i_1, \dots, i_k=1}^n x_{i_1}^1 \cdots x_{i_k}^k c_{i_1, \dots, i_k}(b). \quad (3.2)$$

Then, as

$$\frac{1}{\|z\|} (D^k v(b+z)(x^1, \dots, x^k) - D^k v(b)(x^1, \dots, x^k) - D^{k+1} v(b)(x^1, \dots, x^k, z)) \xrightarrow{z \rightarrow 0} 0$$

we have

$$\phi \left( \frac{1}{\|z\|} \left( \sum_{i_1, \dots, i_k=1}^n x_{i_1}^1 \cdots x_{i_k}^k \left( (c_{i_1, \dots, i_k}(b+z) - c_{i_1, \dots, i_k}(b)) - \sum_{i_{k+1}=1}^n z_{i_{k+1}} c_{i_1, \dots, i_k, i_{k+1}}(b) \right) \right) \right) \xrightarrow{z \rightarrow 0} 0.$$

By lemma 3.3 we infer that  $c_{i_1, \dots, i_k}(\cdot)$  are differentiable and

$$Dc_{i_1, \dots, i_k}(\cdot)(z) = \sum_{i=1}^n z_i c_{i_1, \dots, i_k, i}(b).$$

Thus the condition (3.2) is true also for  $k+1$ . Induction shows that  $f$  is  $2+t$  times differentiable. Arguing as previously (see equation (3.1)) we show that  $f \in C^{2+t}(U)$ . The same argument shows that if  $v \in C^{2+t}(\overline{U})$  then all derivatives of  $f$ , up to order  $2+t$ , extend continuously to  $\overline{U}$ .

The function  $f$  extends to  $\overline{U}$ , since its derivative does. Indeed, let  $b_0 \in \overline{U} \setminus U$  and let a sequence  $(b_n)_{n=1}^\infty \subset U$  converge to  $b_0$ . Then  $(f(b_n))_{n=1}^\infty$  is a Cauchy sequence. Indeed, for curve  $\gamma \subset U$  connecting  $b_n$  and  $b_m$  which satisfies  $|\gamma| \leq h(\|b_n - b_m\|)$ , we have

$$\|f(b_m) - f(b_n)\| = \left\| \int_0^1 Df(\gamma(t))(\gamma'(t)) dt \right\| \leq M|\gamma| \leq Mh(\|b_n - b_m\|),$$

<sup>1</sup>  $(x_1, \dots, \hat{x}_l, \dots, \hat{x}_p, \dots, x_k)$  denotes  $(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_{p-1}, x_{p+1}, \dots, x_k)$ .

as  $Df(\cdot)$  remains bounded by some  $M$  as  $n, m \rightarrow \infty$ . Thus we may define

$$f(b_0) = \lim_{n \rightarrow \infty} f(b_n).$$

This definition does not depend on the choice of the sequence  $(b_n)_{n=1}^\infty$ . Indeed, choose a sequence  $(c_n)_{n=1}^\infty \subset U$  converging to  $b_0$ . Then, as before,

$$\|f(c_n) - f(b_n)\| \leq Mh(\|c_n - b_n\|).$$

Thus

$$\lim_{n \rightarrow \infty} f(b_n) = \lim_{n \rightarrow \infty} f(c_n).$$

In particular, for any sequence  $(d_n)_{n=1}^\infty \subset U$  converging to  $b_0$ , we have

$$f(b_0) = \lim_{n \rightarrow \infty} f(d_n).$$

Thus  $f$  is continuous on  $\overline{U}$ . □

**Remark 3.10.** The assumption about short-path connectedness is used only to show that  $f$  itself (and not its derivatives) extends continuously on  $\overline{U}$ .

**Remark 3.11.** We have actually shown that  $v = \phi(f)$ , where  $\phi$  is a linear functional making the algebra Frobenius.

*Proof of theorem 3.8.* Let  $v: U \rightarrow \mathbb{F}$  be as in theorem, of class  $C^{2+t}$ . Let  $\eta: A^n \rightarrow B$  be a surjective  $A$ -linear map. Consider the composition

$$v \circ \eta: \eta^{-1}(U) \rightarrow \mathbb{F}.$$

Then  $v \circ \eta$  satisfies the assumptions of the lemma 3.9. Indeed,

$$\begin{aligned} D^2(v \circ \eta)(b)(ax, y) &= D(D(v \circ \eta)(b)(ax))(y) = D(Dv(\eta(b))(\eta(ax)))(y) = \\ &= D^2v(\eta(b))(a\eta(x), \eta(y)) = D^2v(\eta(b))(\eta(x), a\eta(y)) = \\ &= D^2(v \circ \eta)(b)(x, ay). \end{aligned}$$

Thus there exist  $A$ -differentiable function  $f: \eta^{-1}(U) \rightarrow A$  in  $C^{2+t}(\eta^{-1}(U))$ , such that  $v \circ \eta = \phi(f)$ .

We shall show that there exists  $g: U \rightarrow A$  such that  $g \circ \eta = f$ . Observe that

$$\phi(aDf(b)(x)) = Dv(\eta(b))(\eta(ax)) = Dv(\eta(b))(a\eta(x))$$

for all  $b \in \eta^{-1}(U)$ ,  $x \in A^n$ ,  $a \in A$ . Thus if  $x \in \ker \eta$ , then  $\phi(aDf(b)(x)) = 0$  for all  $a \in A$ . Since  $A$  is Frobenius,  $Df(b)(x) = 0$ .

Choose two points  $b_1, b_2 \in \eta^{-1}(x)$ . Then  $b_2 - b_1 \in \ker \eta$ . For any  $t \in [0, 1]$ ,  $b_1 + t(b_2 - b_1) \in \eta^{-1}(U)$ . Thus

$$f(b_2) - f(b_1) = \int_0^1 Df(b_1 + t(b_2 - b_1))(b_2 - b_1) dt = 0$$

We may thus define  $g$  on  $U$  by  $g(\eta(x)) = f(x)$ . In view of the lemma 2.5, we see that  $g$  is  $A$ -differentiable and of class  $C^{2+t}(U)$ . Moreover

$$v \circ \eta = \phi(f) = \phi(g) \circ \eta.$$

As  $\eta$  is surjective, we have  $v = \phi(g)$ .

Assume now that  $U$  is short-path connected and  $v \in C^{2+t}(\overline{U})$ . Then

$$v \circ \eta \in C^{2+t}(\overline{\eta^{-1}(U)})$$

and  $\eta^{-1}(U)$  is short-path connected. Thus  $f \in C^{2+t}(\overline{\eta^{-1}(U)})$  and, by lemma 2.5,  $g \in C^{2+t}(\overline{U})$ .  $\square$

**Example 3.12.** Equations

$$D^2v(b)(ax, y) = D^2v(b)(x, ay)$$

are called *generalized Laplace equations*. If  $A = \mathbb{C}$ , treated as  $\mathbb{R}$ -algebra, and  $B = A$ , they are equivalent to ordinary Laplace equation in two variables

$$\frac{\partial^2 v}{\partial y^2}(b) = D^2v(b)(i, i) = D^2v(b)(i^2, 1) = -\frac{\partial^2 v}{\partial x^2}(b).$$

In this case, the theorem tells us that every harmonic function on  $U \subset \mathbb{R}^2$  is a real part of some holomorphic function. From this example one readily sees that the assumption about simple connectedness of  $U$  can not be dropped. Indeed,  $\log|\cdot|$  is harmonic on  $\mathbb{C} \setminus \{0\}$ , but it is not real part of a holomorphic function.

If  $A = \mathbb{C}$  and  $B = \mathbb{C}^n$ , then these equations yield conditions of pluriharmonicity

$$\begin{aligned} \frac{\partial^2 v}{\partial x_k \partial x_l}(b) &= -D^2v(b)(i^2 e_k, e_l) = -D^2v(b)(ie_k, ie_l) = -\frac{\partial^2 v}{\partial y_k \partial y_l}(b), \\ \frac{\partial^2 v}{\partial x_k \partial y_l}(b) &= -D^2v(b)(e_k, ie_l) = -D^2v(b)(ie_k, e_l) = -\frac{\partial^2 v}{\partial y_k \partial x_l}(b). \end{aligned}$$

#### 4. $A$ -analiticity and further properties

**Definition 4.1.** Let  $B$  be finitely generated  $A$ -module. We say that a function  $L: B^i \rightarrow A$  is *symmetric* if

$$L(x_1, \dots, x_i) = L(x_{\sigma(1)}, \dots, x_{\sigma(i)})$$

for any permutation  $\sigma \in S_i$ . We say that a function  $f: B \rightarrow A$  is  *$A$ -polynomial* if  $f(x) = \sum_{i=0}^{\infty} L_i(x, \dots, x)$  for some  $A$ -multilinear, symmetric functions  $L_i: B^i \rightarrow A$  of which only finite number is non-zero. The set of such  $A$ -polynomials is denoted by  $B^{*\infty}$ .

For  $L_i(x, \dots, x)$  we will simply write  $L_i(x^i)$ .

**Definition 4.2.** Let  $U \subset B$  be an open subset. Let  $f: U \rightarrow A$ . We call  $f$  an  $A$ -analytic function if for every point in  $b_0 \in U$  there exist an open neighbourhood  $V \subset U$  of  $b_0$ , such that for  $b \in V$

$$f(b) = \sum_{i=0}^{\infty} L_i((b - b_0)^i),$$

for some symmetric  $A$ -multilinear  $L_i: B^i \rightarrow A$  such that for  $b \in V$

$$\sum_{i=0}^{\infty} \|L_i((b - b_0)^i)\| < \infty.$$

**Example 4.3.** If  $B$  is a free module then any  $A$ -multilinear symmetric mapping  $L_i: B^i \rightarrow A$  is given by

$$L_i(x^1, \dots, x^i) = L_i\left(\sum_{j_1=1}^n x_{j_1}^1 b_{j_1}, \dots, \sum_{j_i=1}^n x_{j_i}^i b_{j_i}\right) = \sum_{j_1, \dots, j_i=1}^n x_{j_1}^1 \cdots x_{j_i}^i c_{j_1, \dots, j_i},$$

for some  $c_{j_1, \dots, j_i} \in A$ ,  $j_1, \dots, j_i = 1, \dots, n$  such that  $c_{j_1, \dots, j_i} = c_{j_{\sigma(1)}, \dots, j_{\sigma(i)}}$  for any permutation  $\sigma \in S_i$ . If  $T: B \rightarrow A$  is a polynomial in coordinate functions  $x_k$ , with coefficients in  $A$ , then

$$T(x) = \sum_{i=1}^l \sum_{0 \leq j_1 \leq \dots \leq j_i \leq n} x_{j_1} \cdots x_{j_i} a_{j_1, \dots, j_i},$$

for some  $a_{j_1, \dots, j_i} \in A$ ,  $0 \leq j_1 \leq \dots \leq j_i \leq n$ . Put

$$L_i(x^1, \dots, x^i) = \sum_{\sigma \in S_i} \frac{1}{i!} \sum_{0 \leq j_1 \leq \dots \leq j_i \leq n} x_{j_1}^{\sigma(1)} \cdots x_{j_i}^{\sigma(i)} a_{j_1, \dots, j_i}.$$

Then  $L_i$  are  $A$ -multilinear and symmetric. Moreover

$$T(x) = \sum_{i=1}^l L_i(x, \dots, x).$$

Thus, in this case, the set of polynomials in coordinate functions  $x_k$ , with coefficients in  $A$ , is equal to  $B^{*\infty}$ .

**Remark 4.4.** Consider the set  $B^*$  of  $A$ -linear functions on  $B$  with values in  $A$ . Suppose that  $g_1, \dots, g_l$  generate  $B^*$  over  $A$ . Any map  $f: B \rightarrow A$  of the form

$$f(x) = \sum_{k_1, \dots, k_l=1}^n a_{k_1, \dots, k_l} g_1(x)^{k_1} \cdots g_l(x)^{k_l}$$

is  $A$ -polynomial.

If  $B$  is a free module, then the previous example shows, that any  $A$ -polynomial map is of this form.

However, if  $B$  is not free, then simple examples show that it is not generally true.

In [1] it is proved that, in the case of a free module,  $A$ -analytic functions on an open set are exactly  $A$ -differentiable real-analytic functions. We prove that this result actually extends to all finitely generated  $A$ -modules.

**Theorem 4.5** (Taylor's theorem). *Let  $f: U \rightarrow A$  be differentiable function of class  $C^{t+1}$ . Let  $x_0, x \in U$  be such that  $U$  contains the segment  $[x_0, x]$ . Then*

$$f(x) = \sum_{k=0}^t \frac{1}{k!} D^k f(x_0)((x-x_0)^k) + \int_0^1 \frac{(1-s)^t}{(t+1)!} D^{t+1} f(x_0 + s(x-x_0))((x-x_0)^{t+1}) ds.$$

**Lemma 4.6** (cf. [8]). *Let  $U \subset \mathbb{F}^t$ . Let  $f: U \rightarrow \mathbb{F}$  be a smooth function. The following conditions are equivalent*

- (i)  $f$  is  $\mathbb{F}$ -analytic,
- (ii) for any  $x \in U$  there exists  $r > 0$  and  $M, C > 0$  such that if  $|y_i - x_i| < r$  then

$$\left| \frac{1}{k!} D^k f(y)(z^k) \right| \leq M C^k r^{-k} \|z\|^k$$

for all  $z \in \mathbb{F}^t$ ,  $k \in \mathbb{N}$ .

**Theorem 4.7.** *Let  $U \subset B$  be an open subset of  $A$ -module  $B$ . Let  $f: U \rightarrow A$ . Then  $f$  is  $A$ -differentiable and  $\mathbb{F}$ -analytic if and only if it is  $A$ -analytic.*

*Proof.* We infer as in [1]. Assume that  $f$  is  $A$ -differentiable and  $\mathbb{F}$ -analytic. Choose  $b_0 \in U$ . By lemma 4.6 applied to each  $f_i$ , there exists  $r > 0$  such that if  $\|b - b_0\| < r$  then

$$\|D^k f(b)(y^k)\| \leq M C^k \|y\|^k$$

for all  $k \in \mathbb{N}$  and some constants  $M, C > 0$ . We bound the remainder term in Taylor's formula

$$\begin{aligned} \|f(y) - \sum_{k=0}^l \frac{1}{k!} D^k f(x)((y-x)^k)\| &\leq \\ &\leq \int_0^1 \frac{(1-s)^l}{(l+1)!} \|D^{l+1} f(x + s(y-x))((y-x)^{l+1})\| ds \leq \frac{1}{l+1} M (C\|y-x\|)^{l+1}. \end{aligned}$$

Thus if  $\|y-x\| < \frac{1}{C}$ , then the series converges. Since  $\frac{1}{k!} D^k f(x)$  is  $A$ -multilinear,  $f$  is  $A$ -analytic.

Assume conversely, that  $f$  is  $A$ -analytic. Then  $A$ -differentiability follows readily from the definition of  $A$ -analyticity. Moreover, for any linear functional  $\psi$  the composition  $\psi(f)$  is  $\mathbb{F}$ -analytic, by example 4.3. This means that  $f$  itself is  $\mathbb{F}$ -analytic.  $\square$

Let us recall some basics of algebra.

**Definition 4.8.** Algebra  $A$  is *local* if it has exactly one maximal ideal.

**Theorem 4.9** (see [3]). *Let  $A$  be a commutative finite dimensional algebra over field  $\mathbb{F}$ . Then there are finitely many ideals  $A_i \subset A$  such that  $A = \bigoplus_{i=1}^n A_i$ . Each  $A_i$  is a local algebra.*



Let  $A = \bigoplus_{i=1}^m A_i$ . Then the unit  $e \in A$  decomposes into a sum

$$e = \sum_{i=1}^m e_i,$$

where  $e_i \in A_i$ . Moreover  $e_i e_j = 0$  if  $i \neq j$ . Since  $e_i a_i = a_i$  for  $a_i \in A_i$ ,  $e_i$  is a unit in  $A_i$ . Since  $e a_i = e_i a_i$ ,  $A_i = e_i A$ .

For any  $A$ -module  $B$  we have  $B = \bigoplus_{i=1}^m e_i B$ . Each  $B_i = e_i B$  is  $A_i$ -module. Define projection  $\pi_{A_i}: A \rightarrow A$  by  $\pi_{A_i}(a) = e_i a$  and  $\pi_{B_i}: B \rightarrow B$  by  $\pi_{B_i}(b) = e_i b$ .

Let  $L: B \rightarrow A$  be  $A$ -linear function. Then

$$L(b) = \sum_{i=1}^m e_i L(b) = \sum_{i=1}^m e_i L(e_i b) = \sum_{i=1}^m \pi_{A_i}(L(\pi_{B_i}(b))).$$

Functions  $\pi_{A_i} \circ L: B_i \rightarrow A_i$  are  $A_i$ -linear.

**Proposition 4.10.** *Assume that  $A = \bigoplus_{i=1}^m A_i$  is a direct product of algebras. Let  $U$  be convex and open set in a finitely generated  $A$ -module  $B$ . Let  $f: U \rightarrow B$  be  $A$ -differentiable. Then*

$$f = \sum_{i=1}^m f_i \circ \pi_{B_i} \quad (4.1)$$

for some  $A_i$ -differentiable functions  $f_i: \pi_{B_i}(U) \rightarrow A_i$ . Conversely, any function of this form is  $A$ -differentiable.

*Proof.* Let  $e \in A$  be the unit. Then, as above,  $e = \sum_{i=1}^m e_i$  for  $e_i \in A_i$  being units of  $A_i$ .

Define  $g_i = e_i f$ . If, for some  $x \in B$ ,  $e_i x = 0$  then

$$Dg_i(y)(x) = D(e_i f)(y)(x) = D(e_i f)(y)(e_i x) = 0$$

since  $e_i^2 = e_i$ . By convexity  $g_i(x) = g_i(y)$  if  $\pi_{B_i}(x) = \pi_{B_i}(y)$ . We define function

$$f_i: \pi_{B_i}(U) \rightarrow A_i$$

by the formula  $f_i(x) = g_i(t)$ , where  $t \in \pi_i^{-1}(x)$ . That is,  $f_i \circ \pi_{B_i} = g_i$ . Lemma 2.5 implies that  $f_i$  are  $A_i$ -differentiable. It follows that

$$f(x) = \sum_{i=1}^m e_i f(x) = \sum_{i=1}^m f_i(\pi_{B_i}(x)).$$

To prove the converse, observe that the derivative of a function of the form (4.1), is given by

$$Df(u)(x) = \sum_{i=1}^m Df_i(\pi_{B_i}(u))(\pi_{B_i}(x))$$

so

$$Df(u)(ax) = \sum_{i=1}^m Df_i(\pi_{B_i}(u))(e_i ax) = \sum_{i=1}^m e_i a Df_i(\pi_{B_i}(u))(e_i x) = a Df(u)(x).$$

Thus  $f$  is  $A$ -differentiable. □

**Lemma 4.11** (see [1]). *Let  $A$  be a finite dimensional commutative local algebra over  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ . Then  $A = A/\mathfrak{m} \oplus \mathfrak{m}$  and  $\pi_A: A \rightarrow A/\mathfrak{m}$  is equal to identity on the first summand and zero on the second.*

As is shown in [1], if there exists maximal ideal  $\mathfrak{m}$  in  $A$ , such that  $A/\mathfrak{m} \cong \mathbb{R}$ , then there exists a smooth  $A$ -differentiable function on a free module which is not  $A$ -analytic. Actually, this result holds for arbitrary modules over Frobenius algebras.

On the other hand, if  $A/\mathfrak{m} \cong \mathbb{C}$  for all maximal ideals, then  $A$  is  $\mathbb{C}$ -algebra (see lemma 4.11) and from proposition 2.11, one infers that any  $A$ -continuous function is  $A$ -analytic.

**Theorem 4.12.** *Let  $A$  be a  $\mathbb{C}$ -algebra. Then every  $A$ -continuous function is  $A$ -analytic.*

*Proof.* Let  $f: U \rightarrow B$  be an  $A$ -continuous function. Then, by proposition 2.11, for any  $b \in B$  and any  $j = 1, \dots, n$ , there exists  $A$ -differentiable function  $g$  such that  $Dg(y)(x) = xf \circ \eta \circ t_j^b(y)$ . Since  $\mathbb{C} \subset A$ , the derivative  $Dg(y)$  is  $\mathbb{C}$ -linear, which means that  $g$  is  $\mathbb{C}$ -differentiable and hence  $\mathbb{C}$ -analytic. Thus  $f \circ \eta \circ t_j^b$  is smooth and  $A$ -differentiable. Then  $f \circ \eta$  is smooth and by proposition 2.8,  $f \circ \eta$  is  $A$ -differentiable. By lemma 2.5  $f$  is  $A$ -differentiable and since  $\mathbb{C} \subset A$  it is  $\mathbb{C}$ -analytic. By theorem 4.7, it is  $A$ -analytic.  $\square$

## 5. Banach algebras of $A$ -differentiable functions

For any two  $A$ -differentiable,  $A$ -valued functions  $f, g$ , their product  $fg$  satisfies

$$\begin{aligned} D(fg)(\cdot)(ax) &= fDg(\cdot)(ax) + gDf(\cdot)(ax) = a(fDg(\cdot)(x) + gDf(\cdot)(x)) = \\ &= aD(fg)(\cdot)(x). \end{aligned}$$

Thus it is again  $A$ -differentiable. Note the essential role played in the equation by commutativity of  $A$ . We see that  $A$ -differentiable functions form an algebra and it is natural to equip this algebra with a norm. For this we equip  $A$  with a Banach algebra norm  $|||$ .

**Definition 5.1.** Assume that  $U \subset B$  is open and bounded. Let  $C(\overline{U}, A)$  denote the set of all  $A$ -valued continuous functions on  $\overline{U}$  equipped with the Banach algebra norm

$$\|f\| = \sup_{x \in \overline{U}} \|f(x)\|.$$

We define subsets of  $C(\overline{U}, A)$  as follows:

- (i)  $C_A(\overline{U}, A)$  is the set of all  $A$ -continuous functions on  $U$ ,
- (ii)  $C_A^\infty(\overline{U}, A)$  is the set of all smooth  $A$ -differentiable functions on  $U$ ,
- (iii)  $C_A^\omega(\overline{U}, A)$  is the set of all  $\mathbb{F}$ -analytic,  $A$ -differentiable functions on  $U$ ,
- (iv)  $C_A^k(\overline{U}, A)$  is the set of all  $A$ -differentiable functions on  $U$  of class  $C^k$ , with all derivatives, of order up to  $k$ , continuous up to boundary.

In the space  $C_A^k(\overline{U}, A)$  we introduce a norm given by the formula

$$\|f\|_k = \sup_{x \in \overline{U}} \|f(x)\| + \sum_{i=1}^k \frac{1}{i!} \sup_{x \in \overline{U}} \sup_{\|y\|=1} \|D^i f(x)((y)^i)\|.$$

We leave it as an exercise to the reader to check that this norm makes  $C_A^k(\overline{U}, A)$  a Banach algebra.

Below, we will also write  $C_A^0(\overline{U}, A)$  for  $C_A(\overline{U}, A)$ .

**Proposition 5.2.** *Assume that  $U \subset B$  is an open, convex and bounded set. Then for any natural  $k \geq 0$  the algebra  $C_A^\infty(\overline{U}, A)$  is dense in  $C_A^k(\overline{U}, A)$ .*

*Proof.* Let us choose a function  $f: \overline{U} \rightarrow A$  belonging to  $C_A^k(\overline{U}, A)$ . Recall that this means that  $f$  is sufficiently differentiable and

$$\int_{\gamma} (f \circ \eta \circ t_j^b) dx = 0$$

for all  $j = 1, \dots, n$ , all  $b \in B$  and all smooth closed curves  $\gamma$  in  $(\eta \circ t_j^b)^{-1}(U)$ .

Choose  $b_0 \in U$  and  $0 < \delta < 1$ . Let

$$U_\delta = \{b \in B: (1 - \delta)(b - b_0) + b_0 \in U\}.$$

Then  $\delta \leq \text{dist}(B \setminus U_\delta, U)$ . For any  $0 < \delta < 1$  we define  $f^\delta: \overline{U}_\delta \rightarrow A$  by

$$f^\delta(b) = f((1 - \delta)(b - b_0) + b_0).$$

Then  $f^\delta \in C_A^k(\overline{U}_\delta, A)$  and converge in  $\|\cdot\|_k$  to  $f$  on  $\overline{U}$ . Thus it is enough to approximate every  $f^\delta$  on  $\overline{U}$ .

Multiplying  $f^\delta$  by a smooth bump function equal to one on  $U_{\delta/4}$  and zero on  $U_{3\delta/4}$  we obtain a compactly supported  $C^k$  function  $\tilde{f}^\delta$  defined on whole  $B$ . Again, it is enough to approximate  $\tilde{f}^\delta$ .

Choose a smooth non-negative function  $\theta: B \rightarrow \mathbb{F}$  supported on the unit ball and such that  $\int_B \theta d\lambda = 1$ . Set

$$\theta_\epsilon(x) = \epsilon^t \theta\left(\frac{x}{\epsilon}\right),$$

where  $t = \dim_{\mathbb{R}} B$ . Then, by uniform continuity of  $\tilde{f}^\delta$  and its derivatives,

$$\tilde{f}_\epsilon^\delta = \tilde{f}^\delta * \theta_\epsilon$$

converge in  $\|\cdot\|_k$  on  $B$  to  $\tilde{f}^\delta$  as  $\epsilon$  goes to zero. Moreover  $\tilde{f}_\epsilon^\delta$  are smooth. We shall now prove that every  $\tilde{f}_\epsilon^\delta$  is  $A$ -continuous on  $U$  provided that  $\epsilon$  is sufficiently small, depending on  $\delta$ . For this, let  $\epsilon < C\delta$ , where a constant  $C$  depends only on the choice of norms, and choose a smooth closed curve  $\gamma: [0, 1] \rightarrow (\eta \circ t_j^b)^{-1}(U)$ .

<sup>2</sup> $\lambda$  denotes the Lebesgue measure.

Then we have by Fubini's theorem

$$\begin{aligned}
\int_{\gamma} (\tilde{f}_{\epsilon}^{\delta} \circ \eta \circ t_j^b) dx &= \int_0^1 \int_{\mathbb{R}^t} \epsilon^t \theta\left(\frac{z}{\epsilon}\right) \tilde{f}^{\delta}(\eta(t_j^b(\gamma(t))) - z) d\lambda(z) d\gamma(t) = \\
&= \int_{B(0, \epsilon)} \epsilon^t \theta\left(\frac{z}{\epsilon}\right) \int_0^1 \tilde{f}^{\delta}(\eta(t_j^b(\gamma(t))) - z) d\gamma(t) d\lambda(z) = \\
&= \int_{B(0, \epsilon)} \epsilon^t \theta\left(\frac{z}{\epsilon}\right) \int_{\gamma} (f^{z, \delta} \circ \eta \circ t_j^b) dx d\lambda
\end{aligned}$$

where  $f^{z, \delta}$ , given by the formula  $f^{z, \delta}(a) = f^{\delta}(a - z)$ , is an  $A$ -continuous function, as  $z \in B(0, \epsilon)$ . Thus the integral vanishes. By corollary 2.9 we see that every  $\tilde{f}_{\epsilon}^{\delta}$  is  $A$ -differentiable.  $\square$

**Corollary 5.3.** *If  $A$  is  $\mathbb{C}$ -algebra, then  $C_A^{\omega}(\overline{U}, A)$  is a complex Banach algebra.*

*Proof.* By theorem 4.12, any  $A$ -continuous function is  $A$ -analytic. Thus  $C_A^{\omega}(\overline{U}, A) = C_A(\overline{U}, A)$ .  $\square$

**Definition 5.4.** Assume that  $U \subset A$  is open. We say that a function  $f: U \rightarrow A$  admits  $A$ -primitive if there exists  $g: U \rightarrow A$  such that  $Dg(\cdot)(x) = xf(\cdot)$ .

**Corollary 5.5.** *Assume that  $U \subset A$  is open and simply connected. Then the set of continuous functions which admit  $A$ -primitive form an algebra.*

*Proof.* Proposition 2.11 shows that a continuous function admit  $A$ -primitive if and only if it is  $A$ -continuous. The set  $C_A(\overline{U}, A)$  of  $A$ -continuous functions is an algebra, since it is closure of the set of smooth  $A$ -differentiable functions.  $\square$

It is interesting to compare these two situations: the first when  $A$  is a  $\mathbb{C}$ -algebra and the second when  $A$  is a  $\mathbb{R}$ -algebra. In the first case every  $A$ -differentiable function is  $A$ -analytic. It may be differentiated as many times as desired, as well primitives may be taken as many times as we want to. In the second case, only the taking of primitives is possible.

## 6. Components of $A$ -differentiable functions revisited

We shall now precisely compute the formulas for  $A$ -differentiable functions on finitely generated modules over commutative, finite dimensional algebras.

### 6.1. Preliminaries

Before we concentrate on the introduced Banach algebras, let us present some preliminary lemmas.

**Proposition 6.1** (see [10]). *For any  $x_1, \dots, x_k \in A$*

$$x_1 \cdots x_k = \frac{1}{2^k k!} \sum_{\epsilon_1, \dots, \epsilon_k = \pm 1} \epsilon_1 \cdots \epsilon_k \left( \sum_{i=1}^k \epsilon_i x_i \right)^k.$$

**Corollary 6.2.** *Let  $I \subset A$  and let  $a \in A$ . Assume that  $I^k a \neq 0$ . Then there exists  $v \in I$  such that  $v^k a \neq 0$ .*

*Proof.* Obvious from the preceding lemma.  $\square$

**Lemma 6.3.** *Let  $U \subset \mathbb{F}^n$  and let  $T \subset \mathbb{F}^m$  be an infinite, bounded set. Let  $k, k_1, \dots, k_m \in \mathbb{N}$ . Let*

$$X_{k_1, \dots, k_m} = \{f: \overline{U} \times T \rightarrow A: f(u, t_1, \dots, t_m) = \sum_{i_1=0}^{k_1} \cdots \sum_{i_m=0}^{k_m} f_{i_1 \dots i_m}(u) t_1^{i_1} \cdots t_m^{i_m}, \\ f_{i_1 \dots i_m} \in C^k(\overline{U}, A)\}.$$

Assume that  $(f^p)_{p=0}^\infty$  is a sequence in  $X_{k_1, \dots, k_m}$  such that

$$\sup_{t \in T} \|f^p(\cdot, t) - f(\cdot, t)\|_k \rightarrow 0.$$

Then for any  $i_1 = 0, \dots, k_1, \dots, i_m = 0, \dots, k_m$

$$\|f_{i_1 \dots i_m}^p - f_{i_1 \dots i_m}\|_k \rightarrow 0.$$

*Proof.* The space

$$Y_{k_1, \dots, k_m} = \{f: T \rightarrow A: f(t_1, \dots, t_m) = \sum_{i_1=0}^{k_1} \cdots \sum_{i_m=0}^{k_m} f_{i_1 \dots i_m} t_1^{i_1} \cdots t_m^{i_m}\}$$

is finite dimensional. Thus any two norms are equivalent on  $Y$ . There exists a universal constant  $G > 0$  such that for any  $h \in Y$

$$\sup_{t \in T} \|h(t)\| \geq G \sum_{i_1=0}^{k_1} \cdots \sum_{i_m=0}^{k_m} \|h_{i_1 \dots i_m}\|.$$

Thus for any  $g \in X_{k_1, \dots, k_m}$  and any indices  $i_1, \dots, i_m$

$$\sup_{t \in T} \|g(\cdot, t)\|_k = \sup_{t \in T} \sum_{i=0}^k \frac{1}{i!} \sup_{u \in \overline{U}} \sup_{\|y\|=1} \|D^i g(u, t)((y)^i)\| \geq \frac{1}{k} G \|g_{i_1 \dots i_m}\|_k.$$

Then

$$\sup_{t \in T} \|f^p(\cdot, t) - f(\cdot, t)\|_k \geq \frac{1}{k} G \|f_{i_1 \dots i_m}^p - f_{i_1 \dots i_m}\|_k.$$

$\square$

## 6.2. Structure

By theorem 4.9 and proposition 4.10 we may now solely consider the case of a local algebra. Let  $A$  be a local algebra with maximal ideal  $\mathfrak{m}$ . For any ideal  $I \subset A$  let

$$\pi_I: I \rightarrow I/\mathfrak{m}I$$

denote the quotient map. Let  $\rho_I: I/\mathfrak{m}I \rightarrow I$  be a fixed  $A/\mathfrak{m}$ -linear map such that  $\pi_I \circ \rho_I = \text{id}$ .

Assume that  $A$ -module  $B$  has a decomposition into two submodules such that  $B = C \oplus D$ . Let

$$\pi_D: B \rightarrow D/\mathfrak{m}D$$

be an  $A$ -linear map which is a composition of a projection onto  $D$  along  $C$  and the quotient map. Let

$$\rho_D: D/\mathfrak{m}D \rightarrow B$$

be  $A/\mathfrak{m}$ -linear map which is a composition of a fixed linear injection from  $D/\mathfrak{m}D$  to  $D$  and the inclusion of  $D$  into  $B$  such that  $\pi_D \circ \rho_D = \text{id}$ .

**Definition 6.4.** We say that a module  $B$  and an ideal  $I \subset A$  have the *property (L)* if the following conditions are satisfied:

- (i) there exist a submodules  $C, D \subset B$  such that  $B = C \oplus D$  and a symmetric  $A/\mathfrak{m}$ -multilinear function  $f: B \times \cdots \times B \rightarrow I/\mathfrak{m}I$  admits a lifting to a symmetric  $A$ -multilinear homomorphism  $h: B \times \cdots \times B \rightarrow I$  such that  $\pi_I \circ h = f$  and for any  $b_1, \dots, b_{i-1} \in B$ ,

$$h(b_1, \dots, b_{j-1}, \cdot, b_{j+1}, \dots, b_{i-1})|_C = 0,$$

if and only if  $f$  is *admissible*, that is for any  $b_1, \dots, b_{i-1} \in B$ ,

$$f(b_1, \dots, b_{j-1}, \cdot, b_{j+1}, \dots, b_{i-1})|_C = 0,$$

- (ii) for any  $j \in \mathbb{N}$ , there exists  $A/\mathfrak{m}$ -linear operator which assigns to an admissible function  $f: B \times \cdots \times B \rightarrow I/\mathfrak{m}I$  its lifting  $G_I^j f: B \times \cdots \times B \rightarrow I$ ,
- (iii) for any  $f: D/\mathfrak{m}D \rightarrow I/\mathfrak{m}I$  and  $v \in B$  we have

$$G_I^1(f \circ \pi_D)(v) = \rho_I(f(\pi_D(v))),$$

- (iv) for any admissible  $f: B \times \cdots \times B \rightarrow I/\mathfrak{m}I$  and any  $v_1, \dots, v_{j+1} \in B$

$$G_I^j f(\cdot, \dots, \cdot, v_{j+1})(v_1, \dots, v_j) = G_I^{j+1} f(v_1, \dots, v_j, v_{j+1})$$

**Lemma 6.5.** Let  $I \subset A$  be an ideal. Let  $i_1, \dots, i_k \in I$  give in quotient map a  $A/\mathfrak{m}$ -basis of  $I/\mathfrak{m}I$ . Then  $A$ -module  $B$  and ideal  $I$  have the property (L) with decomposition  $B = C \oplus B$  if and only if for any  $j = 1, \dots, k$  module  $B$  and ideal  $(i_j)$  have the property (L) with decomposition  $B = C \oplus B$ .

*Proof.* Evident from the definition of the property (L). □

**Lemma 6.6.** Let  $B$  be a finitely generated  $A$ -module such that  $B$  and ideal  $I$  have the property (L). Let  $U \subset B$  be open, convex and bounded set. Assume that  $f: \overline{U} \rightarrow I$  is an  $A$ -continuous function. Then there exists unique  $A/\mathfrak{m}$ -continuous function  $g: \overline{\pi_D(U)} \rightarrow I/\mathfrak{m}I$  such that

$$\pi_I \circ f = g \circ \pi_D. \tag{6.1}$$

*Proof.* Assume first that  $f$  is  $A$ -differentiable.

The map  $\pi_I: I \rightarrow I/\mathfrak{m}I$  is  $A$ -linear, hence  $A$ -differentiable. Thus the composition  $\pi_I \circ f$  is  $A$ -differentiable. Let  $u \in U$ ,  $b \in B$  and  $a \in \mathfrak{m}$ . Since we have

$$D(\pi_I \circ f)(u)(b) \in I/\mathfrak{m}I,$$

then

$$D(\pi_I \circ f)(u)(ab) = aD(\pi_I \circ f)(u)(b) = 0.$$

Moreover, since  $D(\pi_I \circ f)(u) = \pi_I \circ Df(u)$ , function  $D(\pi_I \circ f)(u)$  admits a lifting. By the property (L),

$$D(\pi_I \circ f)(u)|_C = 0.$$

Let  $u_1, u_2 \in U$  be such that  $\pi_D(u_1) = \pi_D(u_2)$ . By convexity of  $U$

$$\pi_I \circ f(u_1) = \pi_I \circ f(u_2).$$

Therefore, there is  $g: \pi_D(U) \rightarrow I/\mathfrak{m}I$  such that  $\pi_I \circ f = g \circ \pi_D$ .  $A$ -differentiability of  $g$  follows from lemma 2.5, as  $\pi_D$  is  $A$ -linear surjection. In particular,  $g$  is  $A/\mathfrak{m}$ -differentiable.

If  $f$  is merely  $A$ -continuous, then by proposition 5.2 we may choose a sequence of smooth  $A$ -differentiable functions  $(f_n)_{n=0}^\infty$  converging to  $f$  uniformly. Then, by the formula (6.1), the corresponding functions  $(g_n)_{n=0}^\infty$  also converge. For its limit (6.1) still holds. Uniqueness is a consequence of (6.1).  $\square$

Observe that the assignment  $f \mapsto g$  is  $A/\mathfrak{m}$  linear. Using the lemma we may define the continuous linear operator

$$H_I: C_A(\overline{U}, I) \rightarrow C_{A/\mathfrak{m}}(\overline{\pi_D(U)}, I/\mathfrak{m}I),$$

by the formula  $H_I(f) = g$ .

**Definition 6.7.** Let  $a \in A$ . We define a natural number  $k(a)$  to be the largest  $k \in \mathbb{N}$  such that  $\mathfrak{m}^k a \neq \{0\}$ .

**Lemma 6.8** (see [1]). *There exist a system of elements  $e_1, \dots, e_r \in \mathfrak{m}$  and a set of multi-indices  $M \subset \mathbb{N}^r$ ,  $M \ni (0, \dots, 0)$ , such that:*

- (i) *the cosets of  $e^\alpha = e_1^{\alpha_1} \dots e_r^{\alpha_r}$ ,  $|\alpha| = k$ ,  $\alpha \in M$ , modulo  $\mathfrak{m}^{k+1}$  are a basis of the  $A/\mathfrak{m}$ -vector space  $\mathfrak{m}^k/\mathfrak{m}^{k+1}$  for every  $k \leq l-1$ ,*
- (ii) *the elements  $(e^\alpha)_{\alpha \in M}$  are a basis of  $A$  as a  $A/\mathfrak{m}$ -vector space.*

Let  $l \in \mathbb{N}$  be such that  $\mathfrak{m}^{l+1} = \{0\}$ , but  $\mathfrak{m}^l \neq \{0\}$ . That is  $l = k(1)$ .

We choose  $\rho_{\mathfrak{m}^k}$  so that  $\rho_{\mathfrak{m}^k}([e^\alpha]) = e^\alpha$  for any  $\alpha \in M$  such that  $|\alpha| = k$ .

**Remark 6.9.** Let  $f \in C_A(\overline{U}, \mathfrak{m}^k)$ . Observe that<sup>3</sup>  $H_{\mathfrak{m}^k} f = \sum_{\alpha \in M, |\alpha|=k} f_\alpha [e^\alpha]$  for some functions  $f_\alpha$ , which we shall, abusing notation, call  $H_{(e^\alpha)} f$ . This shall not result in any misconceptions. For if,  $|\alpha| = k$  and  $f \in C_A(\overline{U}, (e^\alpha))$ , then  $H_{\mathfrak{m}^k} f = H_{(e^\alpha)} f$ .

<sup>3</sup> $[a]$  denotes the coset of an element  $a \in A$ .

**Lemma 6.10.** *Let  $k, p \in \mathbb{N}$ . Let  $B$  be a finitely generated  $A$ -module such that  $B$  and  $\mathfrak{m}^k$  have property (L). Let  $U \subset B$  be an open, convex and bounded set. Assume that  $f \in C_A^p(\overline{U}, \mathfrak{m}^k)$ . Then*

$$H_{\mathfrak{m}^k} f \in \bigoplus_{\alpha \in M, |\alpha|=k} C_{A/\mathfrak{m}}^{p+k(e^\alpha)}(\overline{\pi_D(U)}, (e^\alpha)/\mathfrak{m}(e^\alpha)).$$

Moreover, the operator

$$H_{\mathfrak{m}^k}: C_A^p(\overline{U}, \mathfrak{m}^k) \rightarrow \bigoplus_{\alpha \in M, |\alpha|=k} C_{A/\mathfrak{m}}^{p+k(e^\alpha)}(\overline{\pi_D(U)}, (e^\alpha)/\mathfrak{m}(e^\alpha)),$$

is continuous.

*Proof.* Let  $v \in \mathfrak{m}$ . Let  $f \in C_A^p(\overline{U}, \mathfrak{m}^k)$  be smooth in  $U$ . Then for any  $u \in U$  and any sufficiently small  $b \in B$  by Taylor's theorem we have

$$f(u + vb) = f(u) + \sum_{j=1}^{k(a)} \frac{1}{j!} D^j f(u)((vb)^j) = f(u) + \sum_{j=1}^{l-k} \frac{1}{j!} v^j D^j f(u)((b)^j),$$

as by lemma 2.4 the derivatives are  $A$ -multilinear. Observe that  $f = \rho_{\mathfrak{m}^k} \pi_{\mathfrak{m}^k} + h$ , where  $h$  has values in  $\mathfrak{m}^{k+1}$ . Indeed,  $\pi_{\mathfrak{m}^k}(f - \rho_{\mathfrak{m}^k} \pi_{\mathfrak{m}^k} f) = 0$ . Then for any  $j = 1, \dots, l-k$

$$v^j D^j f(u)((b)^j) = v^j D^j(\rho_{\mathfrak{m}^k} \pi_{\mathfrak{m}^k} f)(u)((b)^j) + v^j D^j h(u)((b)^j).$$

By definition of  $H_{\mathfrak{m}^k}$ ,  $\pi_{\mathfrak{m}^k} \circ f = H_{\mathfrak{m}^k} f \circ \pi_D$ . Thus

$$v^j D^j f(u)((b)^j) = v^j \rho_{\mathfrak{m}^k} D^j(H_{\mathfrak{m}^k} f)(\pi_D(u))((\pi_D(b))^j) + v^j D^j h(u)((b)^j).$$

Then

$$v^j D^j f(u)((b)^j) = \sum_{\alpha \in M, |\alpha|=k} v^j e^\alpha D^j(H_{(e^\alpha)} f)(\pi_D(u))((\pi_D(b))^j) + v^j D^j h(u)((b)^j). \quad (6.2)$$

Choose now a function  $g \in C_A^p(\overline{U}, \mathfrak{m}^k)$  and a sequence  $(g_n)_{n=1}^\infty \in C_A^\infty(\overline{U}, \mathfrak{m}^k)$  which converges to  $g$  in the norm  $\|\cdot\|_p$ . Then, by lemma 6.3 for any small  $b \in B$  the corresponding left-hand sides of the equation (6.2) converge on  $\overline{U}$  in the norm  $\|\cdot\|_p$ . So do right-hand sides.

Therefore for any  $v \in \mathfrak{m}$ ,

$$v^j e^\alpha D^j(H_{(e^\alpha)} f)(\pi_D(u))((\pi_D(b))^j)$$

converge. By corollary 6.2 there is  $v \in \mathfrak{m}$  such that  $v^{k(e^\alpha)} e^\alpha \neq 0$ . It follows that  $(H_{(e^\alpha)} g_n)_{n=1}^\infty$  converges in  $\|\cdot\|_{p+k(e^\alpha)}$ . Since  $(H_{(e^\alpha)} g_n)_{n=0}^\infty$  converges to  $H_{(e^\alpha)} g$  uniformly

$$H_{(e^\alpha)} g \in C_{A/\mathfrak{m}}^{p+k(e^\alpha)}(\overline{\pi_D(U)}, (e^\alpha)/\mathfrak{m}(e^\alpha)).$$

□



**Theorem 6.11.** *Let  $p \in \mathbb{N}$ . Let  $a \in A$ . Let  $B$  be a finitely generated  $A$ -module such that  $B$  and  $(a)$  have the property (L). Let  $U \subset B$  be an open, convex and bounded set. There exists a continuous map*

$$T_{(a)}: C_{A/\mathfrak{m}}^{p+k(a)}(\overline{\pi_D(U)}, (a)/\mathfrak{m}(a)) \rightarrow C_A^p(\overline{U}, (a))$$

such that  $H_{(a)} \circ T_{(a)} = \text{id}$ .

Moreover, if  $k(a) = 0$ , then  $T_{(a)} \circ H_{(a)} = \text{id}$ . If  $b \in A$  is such that  $(a) \supset (b)$ ,  $k(a) > k(b)$  and  $B$  and  $(b)$  have the property (L), then  $H_{(a)} \circ T_{(b)} = 0$ .

*Proof.* We want to define the operator

$$T_{(a)}: C_{A/\mathfrak{m}}^{p+k(a)}(\overline{\pi_D(U)}, (a)/\mathfrak{m}(a)) \rightarrow C_A^p(\overline{U}, (a))$$

such that  $H_{(a)} \circ T_{(a)} = \text{id}$ .

Let  $g \in C_{A/\mathfrak{m}}^{p+k(a)}(\overline{\pi_D(U)}, (a)/\mathfrak{m}(a))$ . For any  $i = 1, \dots, k(a)$ , and any  $p \in \overline{\pi_D(U)}$ , the function  $D^i g(p): \pi_D(B)^i \rightarrow (a)/\mathfrak{m}(a)$  is multilinear and so is its composition with  $\pi_D$ ,  $D^i g(p)(\pi_D(\cdot), \dots, \pi_D(\cdot)): B^i \rightarrow (a)/\mathfrak{m}(a)$ . For any fixed  $b_1, \dots, b_{i-1}$ , the function  $D^i g(p)(\pi_D(b_1), \dots, \pi_D(\cdot), \dots, \pi_D(b_{i-1}))$  is linear and vanishes on  $C$ . By the property (L) there exist liftings  $G_{(a)}^i D^i g(p): B^i \rightarrow (a)$ .

Define

$$T_{(a)}g(u) = \rho_{(a)}g(\pi_D(u)) + \sum_{j=1}^{k(a)} \frac{1}{j!} G_{(a)}^j D^j g(\pi_D(u))((u - \rho_D \pi_D(u))^j). \quad (6.3)$$

Since  $g \in C_{A/\mathfrak{m}}^{p+k(a)}(\overline{\pi_D(U)}, (a)/\mathfrak{m}(a))$  we see that  $T_{(a)}g: \overline{U} \rightarrow (a)$  belongs to  $C_A^p(\overline{U}, A)$ . Moreover  $T_{(a)}$  is continuous.

We need to check that  $T_{(a)}g$  is  $A$ -continuous. For this, let us assume first that  $g$  is smooth in  $\pi_D(U)$ . We shall show that  $T_{(a)}g$  is  $A$ -differentiable.

As  $g$  is smooth in  $\pi_D(U)$  it makes sense to define  $G_{(a)}^i D^i g$ , for all  $i \in \mathbb{N}$ , in the same manner as before.

Assume first that  $v \in B$  is such  $\pi_D(v) = 0$ . Then by multilinearity of  $G_{(a)}^j D^j g$  we see that

$$DT_{(a)}g(u)(v) = \sum_{j=1}^{k(a)} \frac{1}{(j-1)!} G_{(a)}^j D^j g(\pi_D(u))((u - \rho_D \pi_D(u))^{j-1}, v).$$

When  $v \in B$  is such  $\rho_D \pi_D(v) = v$ , then

$$DT_{(a)}g(u)(v) = \rho_{(a)}Dg(\pi_D(u))(\pi_D(v)) + \sum_{j=1}^{k(a)} \frac{1}{j!} G_{(a)}^j D^{j+1} g(\pi_D(u))(\cdot, \pi_D(v))((u - \rho_D \pi_D(u))^j).$$

By the compatibility conditions (ii)-(iv) of the property (L) we see that

$$DT_{(a)}g(u)(v) = G_{(a)}^1 Dg(\pi_D(u))(v) + \sum_{j=1}^{k(a)} \frac{1}{j!} G_{(a)}^{j+1} D^{j+1} g(\pi_D(u))((u - \rho_D \pi_D(u))^j, v).$$

Observe that  $v = v - \rho_D \pi_D(v) + \rho_D \pi_D(v)$ . We have  $\rho_D \pi_D(\rho_D \pi_D(v)) = \rho_D \pi_D(v)$  and  $\rho_D \pi_D(v - \rho_D \pi_D(v)) = 0$ . Therefore the derivative of  $T_{(a)}g$  for an arbitrary  $v$  is equal to

$$\begin{aligned} DT_{(a)}g(u)(v) &= \sum_{j=1}^{k(a)} \frac{1}{(j-1)!} G_{(a)}^j D^j g(\pi_D(u)) ((u - \rho_D \pi_D(u))^{j-1}, v - \rho_D \pi_D(v)) + \\ &+ \sum_{j=0}^{k(a)} \frac{1}{j!} G_{(a)}^{j+1} D^{j+1} g(\pi_D(u)) ((u - \rho_D \pi_D(u))^j, \rho_D \pi_D(v)) = \\ &= \sum_{j=0}^{k(a)} \frac{1}{j!} G_{(a)}^{j+1} D^{j+1} g(\pi_D(u)) ((u - \rho_D \pi_D(u))^j, v) - \\ &- \frac{1}{(k(a))!} G_{(a)}^{k(a)+1} D^{k(a)+1} g(\pi_D(u)) ((u - \rho_D \pi_D(u))^j, v - \rho_D \pi_D(v)) \end{aligned}$$

From the definition of  $k(a)$  and  $A$ -multilinearity of  $G_{(a)}^{k(a)+1} D^{k(a)+1} g(\pi_D(u))$  it follows that

$$G_{(a)}^{k(a)+1} D^{k(a)+1} g(\pi_D(u)) ((u - \rho_D \pi_D(u))^{k(a)}, v - \rho_D \pi_D(v)) = 0.$$

Thus the derivative of  $T_{(a)}g$  is  $A$ -linear and, in consequence,  $T_{(a)}g$  is  $A$ -differentiable.

If  $g \in C_{A/\mathfrak{m}}^{p+k(a)}(\overline{\pi_D(U)}, A/\mathfrak{m})$  is not smooth in  $\pi_D(U)$ , using proposition 5.2 we choose a sequence

$$(g_n)_{n=1}^\infty \subset C_{A/\mathfrak{m}}^{p+k(a)}(\overline{\pi_D(U)}, A/\mathfrak{m})$$

of smooth functions converging in  $|||_{p+k(a)}$  to  $g$ . Then  $T_{(a)}g_n$  converges to  $T_{(a)}g$  in  $C^p(\overline{U}, A)$ . Since all  $T_{(a)}g_n$  are  $A$ -continuous, so is  $T_{(a)}g$ .

It is necessary now to show that  $H_{(a)} \circ T_{(a)} = \text{id}$ . For this it is enough to show that

$$\pi_{(a)} \circ T_{(a)}g = g \circ \pi_D.$$

From the equation (6.3) we have

$$\pi_{(a)} \circ T_{(a)}g(u) = \pi_{(a)} \rho_{(a)} g(\pi_D(u)) = g(\pi_D(u)).$$

If  $k(a) = 0$ , then by equation (6.3),  $T_{(a)}f(u) = \rho_{(a)}f(\pi_D(u))$ , so

$$T_{(a)}H_{(a)}g(u) = \rho_{(a)}H_{(a)}g(\pi_D(u)) = \rho_{(a)}\pi_{(a)}g(u) = g(u),$$

as  $\rho_{(a)}\pi_{(a)} = \text{id}$ .

If now  $(b) \subset (a)$  and  $k(b) < k(a)$ , then  $b = za$  for some  $z \in \mathfrak{m}$ . Since  $T_{(b)}g \in (b)$ , we see that  $\pi_{(a)}T_{(b)}g = 0$ . Therefore  $H_{(a)} \circ T_{(b)} = 0$ .  $\square$

**Remark 6.12.** Assume that  $A$  is a local Frobenius algebra and  $B$  is arbitrary  $A$ -module. Then lemma 6.10, for  $k = 0$ , and theorem 6.11, for  $a = 1$ , hold. Therefore, we see that if  $A/\mathfrak{m} = \mathbb{R}$ , then there are  $A$ -differentiable functions on  $B$  which are not smooth and not analytic. If  $A$  is not local, then it is a product of local Frobenius algebras, so also the general case can be deduced. Thus for such algebras there exist  $A$ -differentiable functions which are not  $A$ -analytic.

**Definition 6.13.** Assume that  $B$  is finitely generated  $A$ -module. Let  $M \subset \mathbb{N}^r$  be a set of multi-indices,  $e_1, \dots, e_r \in \mathfrak{m}$  be such as in lemma 6.8. Assume that for any  $\alpha \in M$ ,  $B$  and  $(e^\alpha)$  have the property (L) with decomposition  $B = C_k \oplus D_k$ ,  $k = |\alpha|$ .

Let  $U$  be a convex, open, bounded subset of  $B$ .

Let

$$T: \bigoplus_{k=0}^l \bigoplus_{\alpha \in M, |\alpha|=k} C_{A/\mathfrak{m}}^{p+k(e^\alpha)}(\overline{\pi_{D_k}(U)}, (e^\alpha)/\mathfrak{m}(e^\alpha)) \rightarrow C_A^p(\overline{U}, A)$$

be defined by the formula

$$T((f_\alpha)_{\alpha \in M}) = \sum_{\alpha \in M} T_{(e^\alpha)} f_\alpha.$$

Let

$$H: C_A^p(\overline{U}, A) \rightarrow \bigoplus_{k=0}^l \bigoplus_{\alpha \in M, |\alpha|=k} C_{A/\mathfrak{m}}^{p+k(e^\alpha)}(\overline{\pi_{D_k}(U)}, (e^\alpha)/\mathfrak{m}(e^\alpha))$$

be defined by the formula

$$Hf = (f_\alpha)_{\alpha \in M},$$

where

$$\begin{aligned} f_{(0, \dots, 0)} &= H_{(e^0)}(f), \\ \sum_{\alpha \in M, |\alpha|=k} f_\alpha[e^\alpha] &= H_{\mathfrak{m}^k}(f - \sum_{\beta \in M, |\beta| < k} T_{(e^\beta)} f_\beta), \end{aligned}$$

for  $k = 1, \dots, l$ .

**Remark 6.14.** If  $f \in C_A^p(\overline{U}, \mathfrak{m}^k)$ , then from theorem 6.11 it follows that

$$\sum_{\alpha \in M, |\alpha|=k} H_{\mathfrak{m}^k} T_{(e^\alpha)} f_\alpha[e^\alpha] = \sum_{\alpha \in M, |\alpha|=k} H_{(e^\alpha)} T_{(e^\alpha)} f_\alpha[e^\alpha] = \sum_{\alpha \in M, |\alpha|=k} f_\alpha[e^\alpha]. \quad (6.4)$$

For  $f \in C_A^p(\overline{U}, \mathfrak{m}^l)$  we have

$$\sum_{\alpha \in M, |\alpha|=l} T_{(e^\alpha)} H_{(e^\alpha)} f = f. \quad (6.5)$$

**Theorem 6.15.** Operators  $H$  and  $T$  are isomorphisms of Banach spaces and are inverses of one another.

*Proof.* Observe that by lemma 6.5  $B$  and  $\mathfrak{m}^k$  have the property (L) for any  $k = 0, 1, \dots, l$ . Let us check that  $H$  is well defined. For this we have to show that for  $f \in C_A^p(\overline{U}, A)$  and all  $k = 0, 1, \dots, l$

$$f - \sum_{\beta \in M, |\beta| < k} T_{(e^\beta)} f_\beta \in C_A^p(\overline{U}, \mathfrak{m}^k). \quad (6.6)$$

For  $k = 0$  this is obvious. We shall proceed inductively. Assume that condition (6.6) holds for  $k < l$ . Then functions  $(f_\alpha)_{\alpha \in M, |\alpha|=k}$  are well defined by the formula

$$\sum_{\alpha \in M, |\alpha|=k} f_\alpha[e^\alpha] = H_{\mathfrak{m}^k}(f - \sum_{\beta \in M, |\beta| < k} T_{(e^\beta)} f_\beta).$$

Moreover, by equation (6.4), we have

$$H_{\mathbf{m}^k}(f - \sum_{\beta \in M, |\beta| < k+1} T_{(e^\beta)} f_\beta) = \sum_{\alpha \in M, |\alpha| = k} f_\alpha[e^\alpha] - \sum_{\beta \in M, |\beta| = k} H_{(e^\beta)}(T_{(e^\beta)} f_\beta)[e^\beta] = 0.$$

This completes the induction. We shall now check that  $T$  and  $H$  are mutual inverses. Let  $f \in C_A^p(\overline{U}, A)$  and  $Hf = (f_\alpha)_{\alpha \in M}$ . Then, by the equation (6.5),

$$\sum_{\beta \in M, |\beta| = l} T_{(e^\beta)} f_\beta = \sum_{\beta \in M, |\beta| = l} T_{(e^\beta)} H_{(e^\beta)}(f - \sum_{\alpha \in M, |\alpha| < l} T_{(e^\alpha)} f_\alpha) = f - \sum_{\alpha \in M, |\alpha| < l} T_{(e^\alpha)} f_\alpha.$$

Therefore

$$f = \sum_{\alpha \in M} T_{(e^\alpha)} f_\alpha.$$

That is  $\text{id} = T \circ H$ . Choose now some

$$f \in \bigoplus_{k=0}^l \bigoplus_{\alpha \in M, |\alpha| = k} C_{A/\mathbf{m}}^{p+k(e^\alpha)}(\overline{\pi_{D_k}(U)}, (e^\alpha)/\mathbf{m}(e^\alpha)).$$

Then  $Tf = \sum_{\alpha \in M} T_{(e^\alpha)} f_\alpha$ . By theorem 6.11 we have  $H_{(e^0)}Tf = f_0$ . Suppose now that for  $k < l$  and all  $\alpha \in M$  such that  $|\alpha| \leq k$  we have  $(HTf)_\alpha = f_\alpha$ . Then, by the definition of  $T$ ,

$$Tf - \sum_{\alpha \in M, |\alpha| \leq k} T_{(e^\alpha)}(HTf)_\alpha = \sum_{\beta \in M, |\beta| > k} T_{(e^\beta)} f_\beta.$$

By the definition of  $H$  we have

$$\begin{aligned} \sum_{\gamma \in M, |\gamma| = k+1} (HTf)_\gamma[e^\gamma] &= H_{\mathbf{m}^{k+1}}(Tf - \sum_{\alpha \in M, |\alpha| \leq k} T_{(e^\alpha)}(HTf)_\alpha) = \\ &= \sum_{\beta \in M, |\beta| > k} H_{\mathbf{m}^{k+1}}(T_{(e^\beta)} f_\beta) = \sum_{\beta \in M, |\beta| = k+1} H_{\mathbf{m}^{k+1}}(T_{(e^\beta)} f_\beta) = \sum_{\beta \in M, |\beta| = k+1} f_\beta[e^\beta]. \end{aligned}$$

From induction it follows that  $H \circ T = \text{id}$ . Continuity of  $T, H$  follows from lemma 6.10 and theorem 6.11.  $\square$

**Remark 6.16.** Since  $A/\mathbf{m}$  is a finite dimensional extension of  $\mathbb{F}$ , there are two cases to consider. Either  $A/\mathbf{m}$  is equal to  $\mathbb{R}$  or to  $\mathbb{C}$ . We want to stress the difference which occurs.

In the complex field case, differentiability of a function in

$$C_{A/\mathbf{m}}^{p+k(a)}(\overline{\pi_D(U)}, (a)/\mathbf{m}(a))$$

in open set  $\pi_D(U)$  is automatic for any  $k \geq 0$ . Indeed, the condition of  $A/\mathbf{m}$ -continuity means that a function is complex differentiable. However, the continuity of derivatives on the closure of  $U$  is not automatic.

In the real field case, which appears to be much more interesting, we see that certain components of a function in  $C_A^p(\overline{U}, (a))$  are necessarily of higher differentiability, which is an unexpected phenomenon.

## 7. General algebra, free module

We shall now consider the case when  $A$  is an arbitrary finite dimensional, commutative algebra and module  $B$  is free. We may, without loss of generality, assume that  $A$  is local. The maximal ideal in  $A$  is denoted by  $\mathfrak{m}$ .

Let  $b_1, \dots, b_n$  be a  $A$ -basis of  $B$ . Define  $\rho_B: B/\mathfrak{m}B \rightarrow B$  by the formula

$$\rho_D\left(\sum_{i=1}^n [a_i] \pi_B(b_i)\right) = \sum_{i=1}^n [a_i] b_i,$$

for any  $a_i \in A$ ,  $i = 1, \dots, n$ .

**Lemma 7.1.** *Assume that  $B$  is a finitely generated, free  $A$ -module. Then that for any  $a \in A$ ,  $B$  and  $(a)$  have property (L) with decomposition  $B = C \oplus B$ ,  $C = \{0\}$ .*

*Proof.* Let  $a \in A$ . We have to show that any symmetric  $A/\mathfrak{m}$ -multilinear map  $f: B \times \dots \times B \rightarrow (a)/\mathfrak{m}(a)$  admits a lifting to an  $A$ -multilinear symmetric map  $h: B \times \dots \times B \rightarrow (a)$  such that  $\pi_{(a)} \circ h = f$ . Let  $f: B \times \dots \times B \rightarrow (a)/\mathfrak{m}(a)$  be symmetric and  $A/\mathfrak{m}$ -multilinear. Define

$$G_{(a)}^m\left(\sum_{i_1=1}^n a_{i_1 1} b_{i_1}, \dots, \sum_{i_m=1}^n a_{i_m m} b_{i_m}\right) = \sum_{i_1=1, \dots, i_m=1}^n a_{i_1 1} \dots a_{i_m m} \rho_{(a)} f(b_{i_1}, \dots, b_{i_m}).$$

Then conditions (i)-(ii) and (iv) from definition 6.4 are clearly satisfied. Let us check the condition (iii). Let  $f: B \rightarrow (a)/\mathfrak{m}(a)$  be  $A/\mathfrak{m}$ -linear. Recall that  $\pi_D = \pi_B$  is a quotient map onto  $B/\mathfrak{m}B$  and  $\rho_D = \rho_B$  is  $A/\mathfrak{m}$ -linear map from  $B/\mathfrak{m}B$  to  $B$ . We have to check that

$$G_{(a)}^1\left(\sum_{i=1}^n [a_i] b_i\right) = \rho_{(a)} f\left(\sum_{i=1}^n [a_i] b_i\right).$$

This is again clear from the definition of  $G_{(a)}^1$ . □

**Theorem 7.2.** *Assume that  $B$  is finitely generated, free module over local algebra  $A$ . Assume that  $U \subset B$  is open, convex and bounded. Then any  $A$ -continuous function in  $f \in C_A^p(\overline{U}, A)$  may be written in the form*

$$f = \sum_{\alpha \in M} \left( \rho_{(e^\alpha)} f_\alpha(\pi_B(u)) + \sum_{j=1}^{k(e^\alpha)} \frac{1}{j!} G_{(e^\alpha)}^j D^j f_\alpha(\pi_B(u)) ((u - \rho_B \pi_B(u))^j) \right).$$

for some functions  $(f_\alpha)_{\alpha \in M} \in \bigoplus_{\alpha \in M} C_{A/\mathfrak{m}}^{p+k(e^\alpha)}(\overline{\pi_B(U)}, (e^\alpha)/\mathfrak{m}(e^\alpha))$ . Conversely, any such function belongs to  $C_A^p(\overline{U}, A)$ . This assignment is an isomorphism of Banach spaces.

*Proof.* Follows directly from lemma 7.1 and theorem 6.15. □

This theorem answers the question raised by Waterhouse in [14] about rules satisfied by  $A$ -differentiable functions on algebras. The case of arbitrary finitely

generated module remains open, but it can be expected that the structure of  $A$ -differentiable functions should rely on the lifting properties of  $A$ -module homomorphisms. Another possible approach is to follow the observation from lemma 2.5 that if  $f \in C_A^p(\overline{U}, A)$ , and  $\eta: A^n \rightarrow B$  is surjective  $A$ -linear map, then  $f \circ \eta$  is in  $C_A^p(\overline{\eta^{-1}(U)}, A)$ . Thus theorem 7.2 gives us a description of  $f \circ \eta$ .

## 8. Algebra generated by one element

We shall now study the case when  $A$  is generated by one element. It is particularly easy and will lead us to interesting applications. Let us first describe how such algebras look and how modules over such algebras look.

Any finite dimensional, commutative algebra over  $\mathbb{F}$  that is generated by one element is isomorphic to  $\mathbb{F}[x]/(P(x))$ , for some polynomial  $P$ .

If  $\mathbb{F} = \mathbb{R}$ , then  $P$  factors into a product of polynomials of the form<sup>4</sup>

$$Q_{\lambda,k}(x) = (x - \lambda)^k, \quad (8.1)$$

for some  $\lambda \in \mathbb{R}$  or of the form

$$R_{\alpha,\beta,k}(x) = (x^2 - 2\alpha x + \beta)^k, \quad (8.2)$$

for some  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha^2 + \beta > 0$ . That is

$$P(x) = \prod_{i=1}^m Q_{\lambda_i, l_i}(x) \prod_{i=1}^n R_{\alpha_i, \beta_i, k_i}(x).$$

Sending  $x \mapsto z$  we have  $\mathbb{R}[x]/(R_{\alpha_i, \beta_i, k_i}(x)) \cong \mathbb{C}[z]/(Q_{\gamma_i, k_i}(z))$ , where  $\gamma_i = \alpha_i \pm \sqrt{\alpha_i^2 + \beta_i}i$ .

By the Chinese remainder theorem we obtain a decomposition of  $\mathbb{R}[x]/(P(x))$  into a direct sum of local algebras

$$\mathbb{R}[x]/(P(x)) \cong \bigoplus_{i=0}^m \mathbb{R}[x]/(Q_{\lambda_i, l_i}(x)) \oplus \bigoplus_{i=0}^n \mathbb{C}[z]/(Q_{\gamma_i, k_i}(z)). \quad (8.3)$$

When  $\mathbb{F} = \mathbb{C}$ , then the polynomial  $P$  factors into a product of polynomials of the form 8.1. Thus

$$\mathbb{C}[x]/(P(x)) \cong \bigoplus_{i=1}^n \mathbb{C}[z]/(Q_{\gamma_i, k_i}(z)). \quad (8.4)$$

Proposition 4.10 tells us that we may restrict ourselves to the case  $A = \mathbb{F}[x]/(Q_{\lambda, l}(x))$ . Let  $e = Q_{\lambda, 1}(x)$ . Then  $e$  generates the maximal ideal  $\mathfrak{m} = (Q_{\lambda, 1}(x))$ . Note that  $\mathfrak{m}^{l-1} \neq \{0\}$  and  $l-1$  is maximal among all such natural numbers. Moreover powers of  $e$ , including  $1 = e^0$ , are a  $A/\mathfrak{m}$ -basis of  $A$ .

From the structure theorem for finitely generated modules over principal ideal domains (see [3]) or from the Jordan canonical form of a matrix, any finitely generated module over such algebra is of the form

$$B = \bigoplus_{i=1}^{j_1} \mathbb{F}[x]/(Q_{\lambda, 1}(x)) \oplus \bigoplus_{i=1}^{j_2} \mathbb{F}[x]/(Q_{\lambda, 2}(x)) \oplus \cdots \oplus \bigoplus_{i=1}^{j_l} \mathbb{F}[x]/(Q_{\lambda, l}(x)),$$

<sup>4</sup>We shall use this notation also for  $\lambda \in \mathbb{C}$ .

for some natural numbers  $j_1, \dots, j_l \geq 0$ ; if  $j_t = 0$  for some  $t = 1, \dots, l$ , then the corresponding summand is omitted.

Let  $e_t$  denote the unit in the summand  $\mathbb{F}[x]/(Q_{\lambda,t}(x))$ . Let for  $k = 0, 1, \dots, l-1$

$$C_k = \bigoplus_{i=1}^{j_1} \mathbb{F}[x]/(Q_{\lambda,1}(x)) \oplus \dots \oplus \bigoplus_{i=1}^{j_{l-k-1}} \mathbb{F}[x]/(Q_{\lambda,l+k-1}(x)),$$

and

$$D_k = \bigoplus_{i=1}^{j_{l-k}} \mathbb{F}[x]/(Q_{\lambda,l-k}(x)) \oplus \dots \oplus \bigoplus_{i=1}^{j_l} \mathbb{F}[x]/(Q_{\lambda,l}(x)).$$

Let

$$\rho_{(e^k)}: (e^k)/\mathfrak{m}(e^k) \rightarrow (e^k)$$

be defined by the formula  $\rho_{(e^k)}(a[e^k]) = ae^k$  for  $a \in A/\mathfrak{m}$ . Then  $\pi_{(e^k)} \circ \rho_{(e^k)} = \text{id}$ . Let

$$\rho_{D_k}: D_k/\mathfrak{m}D_k \rightarrow B$$

be defined by the formula  $\rho_{D_k}(a[e_t]) = ae_t$  for  $a \in A/\mathfrak{m}$  and  $t \geq l-k$ . Then  $\pi_{D_k} \circ \rho_{D_k} = \text{id}$ .

**Lemma 8.1.** *Let  $k = 0, \dots, l-1$ . Then module  $B$  and ideal  $(e^k) \subset A$  have the property (L) with decomposition  $B = C_k \oplus D_k$ .*

*Proof.* Assume that for  $A/\mathfrak{m}$ -linear map  $f: B \rightarrow (e^k)/\mathfrak{m}(e^k)$  there exists  $A$ -linear map  $h: B \rightarrow (e^k)$ , such that  $\pi_{(e^k)} \circ h = f$ . Then

$$0 = h(Q_{\lambda,t}(x)e_t) = Q_{\lambda,t}(x)h(e_t).$$

Since  $(Q_{\lambda,t}(x)) = \mathfrak{m}^t$ , we see that  $h(e_t) \in \text{Ann}(\mathfrak{m}^t) = \mathfrak{m}^{l-t}$ . Thus if  $l-t \geq k+1$ , that is  $t \leq l-k-1$ , then  $\pi_{(e^k)} \circ h(e_t) = f(e_t) = 0$ . As  $e_t$  generate  $\mathbb{F}[x]/(Q_{\lambda,t}(x))$ ,  $f$  is zero on whole  $C_k$ .

Assume now that  $f: B \times \dots \times B \rightarrow (e^k)/\mathfrak{m}(e^k)$  is multilinear map such that for any  $b_1, \dots, b_{j-1} \in B$

$$f(b_1, \dots, b_{m-1}, \cdot, b_{m+1}, \dots, b_{j-1})|_{C_k} = 0.$$

Set

$$h(e_{t_1}, \dots, e_{t_j}) = \rho_{(e^k)} f(e_{t_1}, \dots, e_{t_j}) \quad (8.5)$$

for all  $t = 1, \dots, l$  and all  $i = 1, \dots, j_t$ . Extend this definition by  $A$ -linearity

$$h(Q_{\lambda,s_1}(x)e_{t_1}, \dots, Q_{\lambda,s_j}(x)e_{t_j}) = Q_{\lambda,s_1}(x) \dots Q_{\lambda,s_j}(x) \rho_{(e^k)} f(e_{t_1}, \dots, e_{t_j}), \quad (8.6)$$

for all  $s_r < t_r$ ,  $r = 1, \dots, j$ . Since  $Q_{\lambda,t}(x)e_t = 0$ , we must have

$$Q_{\lambda,t}(x) \rho_{(e^k)} f(e_{t_1}, \dots, e_t, \dots, e_{t_j}) = 0.$$

By assumption  $f(e_{t_1}, \dots, e_t, \dots, e_{t_j}) = 0$  if  $t \leq l-k-1$ . Assume that  $t \geq l-k$ . Then  $\rho_{(e^k)} f(e_{t_1}, \dots, e_t, \dots, e_{t_j}) \in \mathfrak{m}^k$  and  $Q_{\lambda,t}(x) \in \mathfrak{m}^t$ , so

$$Q_{\lambda,t}(x) \rho_{(e^k)} f(e_{t_1}, \dots, e_t, \dots, e_{t_j}) \in \mathfrak{m}^{t+k} \subset \mathfrak{m}^l = \{0\}.$$

This shows that  $h$  is well defined.

Furthermore, we have

$$(\pi_{(e^k)} \circ h)(e_{t_1}, \dots, e_{t_j}) = (\pi_{(e^k)} \circ \rho_{(e^k)})f(e_{t_1}, \dots, e_{t_j}) = f(e_{t_1}, \dots, e_{t_j}).$$

Since both sides are  $A$ -linear and  $e_t$  generate  $B$ ,  $\pi_{(e^k)} \circ h = f$ .

We define  $G_{(e^k)}^j f = h$ . Then  $G_{(e^k)}^j$  are linear. Choose a function  $f: B \rightarrow (e^k)/\mathfrak{m}(e^k)$  which vanishes on  $C_k$  and  $v \in B$ . It is necessary now to show that

$$G_{(e^k)}^j f(\rho_{D_k} \pi_{D_k}(v)) = \rho_{(e^k)} f(\rho_{D_k} \pi_{D_k}(v)).$$

This follows immediately from equation (8.5). The condition (iv) from definition 6.4 follows readily - both sides there are linear, so it is enough to check it on a basis. Thus it follows from equation (8.6).  $\square$

**Theorem 8.2.** *Assume that  $B$  is finitely generated module over algebra  $A = \mathbb{F}[x]/(Q_{\lambda,l})$ . Assume that  $U \subset B$  is open, convex and bounded. Then any  $A$ -continuous function in  $f \in C_A^p(\overline{U}, A)$  may be written in the form*

$$f = \sum_{k=0}^{l-1} \left( \rho_{(e^k)} f_k(\pi_{D_k}(u)) + \sum_{j=1}^{l-1-k} \frac{1}{j!} G_{(e^k)}^j D^j f_k(\pi_{D_k}(u)) ((u - \rho_{D_k} \pi_{D_k}(u))^j) \right).$$

for some functions

$$(f_k)_{k=0,1,\dots,l-1} \in \bigoplus_{k=0}^{l-1} C_{A/\mathfrak{m}}^{p+l-1-k}(\overline{\pi_{D_k}(U)}, (e^k)/\mathfrak{m}(e^k)).$$

Conversely, any such function belongs to  $C_A^p(\overline{U}, A)$ . This assignment is an isomorphism of Banach spaces.

*Proof.* Follows readily from lemma 8.1 and theorem 6.15.  $\square$

**Lemma 8.3.** *Let  $A = \mathbb{F}[x]/(Q_{\lambda,l}(x))$ . Let  $\phi: A \rightarrow \mathbb{F}$  be defined by the formula*

$$\phi\left(\sum_{i=0}^{l-1} a_i e^i\right) = a_{l-1},$$

where  $a_i \in \mathbb{F}$ . Assume that  $\overline{\rho_{D_i} \pi_{D_i}(U)} \subset \overline{U}$  for all  $i = 0, \dots, l-1$ . Then for any  $k = 0, \dots, l-1$  and any

$$f \in C_{A/\mathfrak{m}}^{p+l-1-k}(\overline{\pi_{D_k}(U)}, (e^k)/\mathfrak{m}(e^k))$$

we have

$$\phi(e^{l-1-k} T_{(e^k)} f)|_{\overline{\rho_{D_k} \pi_{D_k}(U)}} = f \circ \pi_{D_k}$$

and if  $j \neq k$

$$\phi(e^{l-1-j} T_{(e^k)} f)|_{\overline{\rho_{D_j} \pi_{D_j}(U)}} = 0.$$



If functions

$$f_k \in C_{A/\mathfrak{m}}^{p+l-1-k}(\overline{\pi_{D_k}(U)}, (e^k)/\mathfrak{m}(e^k)),$$

$k = 0, \dots, l-2$ , have derivatives equal to zero and

$$f_{l-1} \in C_{A/\mathfrak{m}}^p(\overline{\pi_{D_{l-1}}(U)}, (e^{l-1})/\mathfrak{m}(e^{l-1}))$$

is equal to zero, then

$$\phi(Tf) = 0.$$

*Proof.* By equation (6.3) it follows that  $T_{(e^k)}f|_{\overline{\rho_{D_k}\pi_{D_k}(U)}} = \rho_{(e^k)}f \circ \pi_{D_k}$ . Then

$$\phi(e_{l-1-k}T_{(e^k)}f)|_{\overline{\rho_{D_k}\pi_{D_k}(U)}} = \phi(e_{l-1-k}f \circ \pi_{D_k}e_k)|_{\overline{\rho_{D_k}\pi_{D_k}(U)}} = f \circ \pi_{D_k}.$$

and if  $j \neq k$

$$\phi(e_{l-1-j}T_{(e^k)}f)|_{\overline{\rho_{D_k}\pi_{D_k}(U)}} = \phi(e_{l-1-j}f \circ \pi_{D_k}e_k)|_{\overline{\rho_{D_k}\pi_{D_k}(U)}} = 0.$$

Let  $f_k \in C_{A/\mathfrak{m}}^{p-k+l-1}(\overline{\rho_{D_k}\pi_{D_k}(U)}, (e^k)/\mathfrak{m}(e^k))$ ,  $k = 0, \dots, l-2$  have derivatives equal to zero and let  $f_{l-1} \in C_{A/\mathfrak{m}}^p(\overline{\rho_{D_{l-1}}\pi_{D_{l-1}}(U)}, (e^{l-1})/\mathfrak{m}(e^{l-1}))$  be equal to zero. Then by equation (6.3) it follows that  $T_{(e^k)}f_k = \rho_{(e^k)}f_k \circ \pi_{D_k}$  for  $k = 0, \dots, l-2$  and  $T_{l-1}f_{l-1} = 0$ . Since  $\phi(e^k) = 0$  for  $k = 0, \dots, l-2$  we have

$$\phi(Tf) = \sum_{k=0}^{l-2} \phi(\rho_{(e^k)}f_k \circ \pi_{D_k}) = 0.$$

□

## 9. The equation $\text{grad}(w) = M\text{grad}(v)$

This section is devoted to application of the theory to the equation  $\text{grad}(w) = M\text{grad}(v)$ . In particular, we study the case, when the matrix  $M$  has at least two Jordan blocks corresponding to the same eigenvalue, which was not covered by Waterhouse in [13].

Our equation can be reformulated in an equivalent form

$$Dw(\cdot)(x) = Dv(\cdot)(M^T x),$$

for all  $x \in \mathbb{F}^n$ .

Observe that an algebra  $A$  generated by a matrix  $M$  is isomorphic to  $\mathbb{F}[x]/(P(x))$ , where  $P$  is the minimal polynomial of  $M$ . By example 3.2  $A$  is a Frobenius algebra.

We prove an analogue of the theorem 5.1 from [13].

**Theorem 9.1.** *Let  $t \geq 0$  be a natural number. Let  $U \subset \mathbb{F}^n$  be an open, simply connected set. Let  $A$  be an algebra generated by matrix  $M^T \in M_{n \times n}(\mathbb{F})$  and let  $B = \mathbb{F}^n$  with the natural structure of  $A$ -module. Let  $v, w: U \rightarrow \mathbb{F}$ . The following conditions are equivalent*

- (i)  $v, w$  are  $C^{2+t}$  functions satisfying  $Dw = Dv \circ M^T$ ,

- (ii)  $v = \phi(f)$  is a component function of an  $A$ -differentiable function  $f: U \rightarrow A$  of class  $C^{2+t}$ , and  $w = \phi(M^\top f) + c$ , where  $c$  is a constant.

Assume additionally that  $U$  is short-path connected<sup>5</sup>. Then the following conditions are equivalent

- (i)  $v, w$  are  $C^{2+t}(\overline{U})$  functions satisfying  $Dw = Dv \circ M^\top$ ,
- (ii)  $v = \phi(f)$  is a component function of an  $A$ -differentiable function  $f: U \rightarrow A$  of class  $C^{2+t}(\overline{U})$ , and  $w = \phi(M^\top f) + c$ , where  $c$  is a constant.

*Proof.* Let us first prove the first equivalence. Assume that  $f$  is  $A$ -differentiable and of class  $C^{2+t}$  and  $v = \phi(f)$  and  $w = \phi(M^\top f) + c$ . Then, thanks to  $A$ -linearity of derivative of  $f$ , we have:

$$Dw(b)(x) = \phi(DM^\top f(b)(x)) = \phi(Df(b)(M^\top x)) = Dv(b)(M^\top x).$$

Conversely, if  $v$  and  $w$  satisfy  $Dw = Dv \circ M^\top$ , then the right-hand side of the equation is a derivative of some  $C^{2+t}$  function. Hence its second derivative must be symmetric. Thus

$$\begin{aligned} D^2v(b)(M^\top x, y) &= D(Dv(b)(M^\top x))(y) = D(Dw(b)(x))(y) = \\ &= D^2w(b)(x, y) = D^2w(b)(y, x) = D^2v(b)(M^\top y, x). \end{aligned}$$

As  $M^\top$  generates the algebra, this condition holds also for any element  $a \in A$ . By theorem 3.8, there exists uniquely determined, up to a constant,  $A$ -differentiable  $f: U \rightarrow A$ , of class  $C^{2+t}$ , which satisfies  $v = \phi(f)$ . Then  $w$  and  $\phi(M^\top f)$  have the same derivative by the first part of the proof, so they differ by a constant.

The second equivalence follows from theorem 3.8 in the same way.  $\square$

**Theorem 9.2.** *The following conditions are equivalent:*

- (i) any functions  $v, w$  of class  $C^2$  which satisfy  $Dw = Dv \circ M^\top$ , are analytic and are components of  $A$ -analytic functions.
- (ii)  $M$  has no real eigenvalues

*Proof.* If  $M$  does not have any real eigenvalue, then algebra  $A$  generated by  $M^\top$  is a  $\mathbb{C}$ -algebra. By theorem 4.12 every  $A$ -differentiable function is  $A$ -analytic. So is a component of such a function.

Conversely, if  $M$  has a real eigenvalue, then by proposition 4.10 and §8 we may assume that it is the only eigenvalue. Then theorem 8.2 gives us examples of  $A$ -differentiable functions which all components are not analytic.  $\square$

## 10. Boundary value problem for generalized Laplace equations

Now we are ready to properly pose boundary value problem. We assume that  $U$  is a open, convex, bounded set. Our aim is to find what a boundary for the equation should be and what conditions we should impose on the boundary values to acquire

<sup>5</sup>See definition 3.7.

a solution of prescribed differentiability that is unique up to a constant. Theorem 9.1 tells us that solving the equation, up to a constant, is the same as finding  $A$ -differentiable function. By theorem 8.2 we see that to describe  $A$ -differentiable function it is sufficient and enough to prescribe some functions on the projections of  $U$ , of sufficient differentiability, and then extend them to the whole of  $U$  by Taylor's formula.

**Example 10.1.** Consider the matrix

$$M = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}.$$

The minimal polynomial of  $M^\top$  is  $P(x) = (x - \lambda)^2$ . Algebra  $A$  generated by  $M^\top$  is then equal to  $\mathbb{R}[x]/(x - \lambda)^2$ .  $A$ -module  $B = \mathbb{R}^3$  has the decomposition

$$B = \mathbb{R}[x]/(x - \lambda) \oplus \mathbb{R}[x]/(x - \lambda)^2.$$

$A$  has a basis  $1, e$ , where  $e = x - \lambda$ .  $B$  has a basis  $(e_1, e_2, e_3)$ , such that  $ee_1 = 0$ ,  $ee_2 = 0$ ,  $ee_3 = e_2$ .  $A$  is a Frobenius algebra, with the functional  $\phi(x_1 1 + x_2 e) = x_2$ ,  $x_1, x_2 \in \mathbb{R}$ . Let

$$U = (0, 1)^3 = \{x_1 e_1 + x_2 e_2 + x_3 e_3 \in B : 0 < x_i < 1\}.$$

Let  $t \in \mathbb{N}$ ,  $v \in C^{2+t}(\overline{U})$ . Consider the equation

$$D^2 v(\cdot)(z, M^\top y) = D^2 v(\cdot)(M^\top z, y), \quad (10.1)$$

for all  $z, y \in B$ . Equivalently  $D^2 v(\cdot)(z, ey) = D^2 v(\cdot)(ez, y)$ . This means that

$$D^2 v(\cdot)(z_1 e_1 + z_2 e_2 + z_3 e_3, y_3 e_2) = D^2 v(\cdot)(z_3 e_2, y_1 e_1 + y_2 e_2 + y_3 e_3),$$

that is

$$\frac{\partial^2 v}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 v}{\partial x_2^2} = 0.$$

Theorem 3.8 tells us that any such  $v$  is given by  $v = \phi(f)$  for some  $A$ -differentiable  $f$  of class  $C^2(\overline{U})$ . Further, by theorem 8.2, any such  $f$  is uniquely determined by two functions -  $f_0$  in  $C_{\mathbb{R}}^3(\pi_{D_0}(U), \mathbb{R})$ , and  $f_1$  in  $C_{\mathbb{R}}^2(\pi_{D_1}(U), \mathbb{R})$ , where

$$\begin{aligned} D_0 &= \mathbb{R}[x]/(x - \lambda)^2 \\ D_1 &= \mathbb{R}[x]/(x - \lambda) \oplus \mathbb{R}[x]/(x - \lambda)^2, \end{aligned}$$

and

$$\begin{aligned} \pi_{D_0} : B &\rightarrow D_0/\mathfrak{m}D_0, & \pi_{D_0}(x_1 e_1 + x_2 e_2 + x_3 e_3) &= x_2[e_2], \\ \pi_{D_1} : B &\rightarrow D_1/\mathfrak{m}D_1, & \pi_{D_1}(x_1 e_1 + x_2 e_2 + x_3 e_3) &= x_1[e_1] + x_2[e_2]. \end{aligned}$$

The extension is given by

$$\begin{aligned} f(u) &= \rho_{(e^0)} f_0(\pi_{D_0}(u)) + G_{(e^0)}^1(Df_0(\pi_{D_0}(u)))(u - \rho_{D_0} \pi_{D_0} u) + \rho_{(e^1)} f_1(\pi_{D_1}(u)) = \\ &= f_0(u_2[e_2])1 + u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2])e + f_1(u_1[e_1] + u_2[e_2])e. \end{aligned}$$

Thus

$$v(u) = \phi(f(u)) = u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2]) + f_1(u_1[e_1] + u_2[e_2]).$$

Therefore any solution to the equation  $\text{grad}(w) = M \text{grad}(v)$  is of the form

$$\begin{aligned} v(u) &= \phi(f(u)) = u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2]) + f_1(u_1[e_1] + u_2[e_2]), \\ w(u) &= \phi(M^\top f(u)) + c = f_0(u_2[e_2]) + \lambda u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2]) + \lambda f_1(u_1[e_1] + u_2[e_2]) + c. \end{aligned}$$

We see that there is unique solution  $v$  of generalized Laplace equations (10.1) such that it has fixed values on  $\overline{\rho_{D_1} \pi_{D_1}(U)}$  and such that  $w - \lambda v$  has fixed, up to a constant, values on  $\rho_{D_0} \pi_{D_0}(U)$ .

The last observation in the above example may be generalized to a theorem that links the components of  $A$ -differentiable functions with the solutions of boundary value problem for generalized Laplace equations.

For clarity, we shall only treat the case when  $M$  has one eigenvalue. A general case may also be inferred from this particular case, using proposition 4.10. However, we shall not state the corresponding theorem, due to its notation complication level.

Assume first that matrix  $M \in M_{n \times n}(\mathbb{R})$  has one eigenvalue  $\lambda \in \mathbb{F}$ ,  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ . Then the algebra  $A$  generated by  $M^\top$  is a local algebra isomorphic to  $\mathbb{F}[x]/(x - \lambda)^l$ . Define  $\phi_{\mathbb{R}}: A \rightarrow \mathbb{R}$  by the formula

$$\phi\left(\sum_{i=0}^{l-1} a_i [(x - \lambda)^i]\right) = a_{l-1}.$$

and  $\phi_{\mathbb{C}}: A \rightarrow \mathbb{R}$  by the formula

$$\phi\left(\sum_{i=0}^{l-1} a_i [(x - \lambda)^i]\right) = \text{Im } a_{l-1}.$$

Then  $\phi_{\mathbb{F}}$  makes  $A$  a Frobenius algebra. Define  $\mu_{\mathbb{F}}: \mathbb{F} \rightarrow \mathbb{R}$  by  $\mu_{\mathbb{R}} = \text{id}$  and  $\mu_{\mathbb{C}} = \text{Im}$ . Recall that  $(x - \lambda)^k = e^k$ , for all  $k = 0, \dots, l - 1$ .

**Theorem 10.2.** *Assume that  $U \subset \mathbb{R}^n$  is convex, open and bounded set. Assume that  $\overline{\rho_{D_i} \pi_{D_i}(U)} \subset \overline{U}$  for all  $i = 0, \dots, l - 1$ . Let  $t \geq 2$ . Then for any functions*

$$f_i \in C_{\mathbb{F}}^{t+l-1-i}(\overline{\pi_{D_i}(U)}, \mathbb{F}), i = 0, \dots, l - 1,$$

*there exists a unique  $v \in C^t(\overline{U})$  such that*

$$\begin{aligned} D^2 v(\cdot)(M^\top x, y) &= D^2 v(\cdot)(x, M^\top y), \\ v|_{\overline{\rho_{D_{l-1}} \pi_{D_{l-1}}(U)}} &= \mu_{\mathbb{F}} f_{l-1} \circ \pi_{D_{l-1}}, \\ Dv(\cdot)((M^\top - \lambda I)^{l-1-i} x)|_{\overline{\rho_{D_i} \pi_{D_i}(U)}} &= \mu_{\mathbb{F}} D(f_i \circ \pi_{D_i})(\cdot)(x), \\ x \in \rho_{D_i} \pi_{D_i}(B), i &= 0, \dots, l - 2. \end{aligned} \tag{10.2}$$

The unique solution is given by  $v = \phi_{\mathbb{F}}(Tf)$ , where

$$Tf = \sum_{k=0}^{l-1} \left( \rho_{(e^k)} f_k(\pi_{D_k}(u)) + \sum_{j=1}^{l-1-k} \frac{1}{j!} G_{(e^k)}^j D^j f_k(\pi_{D_k}(u)) ((u - \rho_{D_k} \pi_{D_k}(u))^j) \right).$$

*Proof.* Let us show that  $v = \phi_{\mathbb{F}}(Tf)$  solves the system. Since  $Tf$  is  $A$ -differentiable, by lemma 3.6 it follows that for any  $x, y \in \mathbb{R}^n$

$$D^2 v(b)(M^{\top} x, y) = D^2 v(b)(x, M^{\top} y).$$

It is necessary to show that the boundary conditions are satisfied. For this we use lemma 8.3 and infer that

$$\phi_{\mathbb{F}}(e^{l-1-i} Tf)|_{\overline{\rho_{D_i} \pi_{D_i}(U)}} = \mu_{\mathbb{F}} f_i \circ \pi_{D_i}$$

and in particular

$$v|_{\overline{\rho_{D_{l-1}} \pi_{D_{l-1}}(U)}} = \mu_{\mathbb{F}} f_{l-1} \circ \pi_{D_{l-1}}.$$

By  $A$ -linearity of derivative of  $Tf$  we have

$$Dv(b)(e^{l-1-i} x) = \phi_{\mathbb{F}}(DTf(b)(e^{l-1-i} x)) = D\phi_{\mathbb{F}}(e^{l-1-i} Tf)(b)(x) = \mu_{\mathbb{F}} D(f_i \circ \pi_{D_i})(b)(x)$$

for all  $b \in \overline{\rho_{D_i} \pi_{D_i}(U)}$  and all  $x \in \rho_{D_i} \pi_{D_i}(B)$ . This means that  $v$  satisfies equations (10.2).

It is necessary to show that the solution is unique. Assume that

$$\begin{aligned} D^2 v(\cdot)(M^{\top} x, y) &= D^2 v(\cdot)(x, M^{\top} y), \\ v|_{\overline{\rho_{D_{l-1}} \pi_{D_{l-1}}(U)}} &= 0, \\ Dv(\cdot)((M^{\top} - \lambda I)^{l-1-i} x)|_{\overline{\rho_{D_i} \pi_{D_i}(U)}} &= 0, \\ x \in \rho_{D_i} \pi_{D_i}(B), i &= 0, \dots, l-2. \end{aligned}$$

It is necessary to show that  $v = 0$ . Since  $U$  is short-path connected, from theorem 3.8 we infer that  $v = \phi_{\mathbb{F}}(h)$  for some  $h \in C_A^t(\overline{U}, A)$ . By theorem 8.2,  $h = Tg$  for some

$$g \in \bigoplus_{k=0}^{l-1} C_{\mathbb{F}}^{t+l-1-k}(\overline{\pi_{D_k}(U)}, \mathbb{F}).$$

The first part of the proof shows that  $\mu_{\mathbb{F}} g_{l-1} = 0$  and for all  $i = 0, \dots, l-2$  functions  $\mu_{\mathbb{F}} g_i$  have zero derivatives. If  $\mathbb{F} = \mathbb{R}$ , then, by lemma 8.3,  $v = \phi_{\mathbb{R}}(Tg) = 0$ .

Assume that  $\mathbb{F} = \mathbb{C}$ . Then, as  $g_i$  are  $\mathbb{C}$ -differentiable, Cauchy-Riemann equations imply that  $g_i$  have zero derivatives for  $i = 0, 1, \dots, l-2$  and that  $g_{l-1}$  is a real constant. Thus  $\phi_{\mathbb{C}}(Tg_{l-1}) = 0$  and lemma 8.3 implies that  $\phi_{\mathbb{C}}(T(g - g_{l-1})) = 0$ . Therefore  $\phi_{\mathbb{C}}(Tg) = 0$ .  $\square$

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